4 BASICS OF PROBABILITY

- **Experiment** is a process of observation that leads to a single outcome that cannot be predicted with certainty.

  Examples:

  1. Pull a card from a deck
  2. Toss a coin
  3. Response time.

- **Sample Space:** All outcomes of an experiment. Usually denoted by $\mathbf{S}$.

- **Event** denoted by $\mathbf{E}$ is any subset of $\mathbf{S}$

  1. E = Spades
  2. E = Head
  3. E = Component is functioning

- $\mathbf{P(E)}$ denotes the probability of the event $\mathbf{E}$.

  1. P(E) = P(Spades)
  2. P(E) = P(Head)
  3. P(E) = P(Component is functioning)

Calculating Probabilities

- **CLASSICAL APPROACH**:

  Assumes all outcomes of the experiment are equally likely:

$$P(E) = \frac{\text{number of favourable cases}}{\text{number of possible cases}}$$

  Example: Roll a fair die.

  E = even number

$$P(E) = \frac{3}{6}$$

- **RELATIVE FREQUENCY APPROACH**:

  Interprets the probability as the relative frequency of the event over a long series of experiment.

$$P(E) = \frac{\text{no. of times } \mathbf{E} \text{ occurs}}{\text{no. of times experiment is repeated}}$$

  Example: Roll a die a large number of times and observe number of times an even number occurs.

$$P(E) = \frac{\text{number of observed evens}}{\text{number of times the die is rolled}}$$

Examples:

1. Toss a coin.
   What is the probability of getting a head?

2. Toss two coins.
   What is the probability of getting at least one head?

3. Select a card from a deck.
   What is the probability that it is a diamond?

4. A group of four integrated-circuit (IC) chips consists of two good chips and two defective chips. If three chips are selected at random from this group, what is the probability that two are defective?
   Solution:
   A natural sample space for this problem consists of all possible three-chip selections from the group of four chips:

   $$S = \{g_1g_2d_1, g_1g_2d_2, g_1d_1d_2, g_2d_1d_2\}.$$

   $E =$ Two of the three selected chips are defective.
   Since the two sample points $g_1d_1d_2$ and $g_2d_1d_2$ are favourable to the event E and since the sample space has four points, we conclude that $P(E) = 2/4 = 1/2$.

Permutations and Combinations:
We have seen that finding $P(E)$ simply involves counting the number of equally likely outcomes favourable to E. However, counting by hand may not be feasible when the sample space is large.

Example
Passwords consist of three letters.
Find the probability that a randomly chosen password will not have any repeated letters.

Let $A = \{a, b, \ldots, z\}$ be the alphabet of 26 letters.
Then the sample space is

$$S = \{(\alpha, \beta, \gamma) : \alpha \in A, \beta \in A, \gamma \in A\}$$

and the event of interest is

$$E = \{(\alpha, \beta, \gamma) : \alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha\}$$

The number of cases favourable to E is
26 for the first letter
25 for the second
24 for the third,
giving a total of $26 \times 25 \times 24 = 15,600$

The number of possible cases in S $= 26^3 = 17,576.$

Then

$$P(E) = \frac{15,600}{17,576} = 0.89.$$

Permutation

The number of ordered sequences where repetition is not allowed, i.e. no element can appear more than once in the sequence.

Examples:

1. Three elements {1,2,3}.
   How many sequences of two elements from these three?
   (1,2); (1,3); (2,1); (2, 3); (3,1); (3,2).
   Six ordered sequences altogether.

2. Four elements {1,2,3,4}.
   How many sequences of two elements from these four?
   (1,2); (1,3); (1,4) (2,1); (2, 3); (2,4);
   (3,1); (3,2); (3,4); (4,1); (4,2); (4,3).
   Twelve ordered sequences altogether.

3. Four elements {1,2,3,4}.
   How many sequences of three elements from these four?
   (1,2,3); (1,3,2); (1,2,4); (1,4,2); (1,3,4); (1,4,3)
   (2,1,3); (2,3,1); (2,1,4); (2,4,1); (2,3,4); (2,4,3)
   (3,1,2); (3,2,1); (3,2,4); (3,4,2); (3,1,4); (3,4,1)
   (4,1,3); (4,3,1); (4,1,2); (4,2,1); (4,3,2); (4,2,3)
   Twenty-four ordered sequences.

Permutations:

Ordered samples (sequences) of size k from n

$$^{n}P_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

1. $^{3}P_2 = 3 * 2 = 6$

   In *R*

   ```
   prod (3:2)
   [1] 6
   ```

2. $^{4}P_2 = 4 * 3 = 12$

   In *R*

   ```
   prod (4:3)
   [1] 2
   ```

3. $^{4}P_3 = 4 * 3 * 2 = 24$

   In *R*

   ```
   prod (4:2)
   [1] 24
   ```

Example

A series of 10 jobs arrive at a computing centre with 10 processors. Assume that each of the jobs is equally likely to go through any of the processors. Find the probability that all processors are occupied.

There are 10^{10} ways of assigning the 10 jobs to the 10 processors.

Let E be the event that all the processors are occupied.

i.e

all the 10 jobs go to different processors.

The number of ways that each processor receives one job is

$$10! = {}^{10}P_{10}$$

$$P(\text{All different processors}) = \frac{{}^{10}P_{10}}{10^{10}}$$

Calculating this in *R* gives

```
 prod(10:1)/10^10
[1] 0.00036288
```

less that .04%

Convesely

$P(\text{At least 2 jobs assigned to at least 1 processors}) = 1 - 0.00036288 = 0.999637$

almost certainty

Combinations:

Example

In how many ways can we select three distinct letters $\{\alpha, \beta, \gamma\}$ from the alphabet $A = \{a, b, \ldots, z\}$?

Taking the three letters a, b, c, the ordered sets are
$(a,\ b,\ c),\ (a,\ c,\ b),\ (b,\ a,\ c),$
$(b,\ c,\ a),\ (c,\ a,\ b),\ (c,\ b,\ a),$

6 permutations in all; but if we ignore the order, there is just one combination.

Combinations:

The number of unordered sets of distinct elements, i.e. repetition is not allowed.

Examples:

1. Three elements $\{1,2,3\}$.
   How many sets (combinations) of two elements from these three?
   Recall for permutations (ordered sequences) there are 6 in all:
   (1,2); (1,3); (2,1); (2, 3); (3,1); (3,2).
   For combinations (1, 2) and (2,1) are equivalent.
   Number of unordered sets $= 3$:
   $\{1,2\}$; $\{1,3\}$; $\{2,3\}$

2. Four elements $\{1,2,3,4\}$.
   How many combinations of two elements from four?
   $\{1,2\}$; $\{1,3\}$; $\{1,4\}$ $\{2,3\}$; $\{2, 4\}$; $\{3,4\}$;
   Six unordered sets altogether.

3. Four elements $\{1,2,3,4\}$.
   How many unordered of three elements from four?
   $\{1,2,3\}$; $\{1,2,4\}$; $\{2,3,4\}$; $\{3,1,4\}$
   Four unordered sequences.

Combinations:
Number of ways of selecting k distinct elements from n or equivalently number of unordered samples of size k, without replacement from n

$$^{n}C_k = \frac{^{n}P_k}{k!} = \frac{n!}{k!(n-k)!}$$

This is the number of combinations of n distinct objects taken k at a time.
Alternative Notation: $\binom{n}{k} = {}^{n}C_k$.

For example:

1. $^{3}C_2 = \frac{^{3}P_2}{2!} = \frac{3*2}{2*1} = 3$

   In *R*

   ```
   choose(3,2)
   [1] 3
   ```

2. $^{4}C_2 = \frac{^{4}P_2}{2!} = \frac{4*3}{2*1} = 6$

   In *R*

   ```
   choose(4,2)
   [1] 6
   ```

3. $^{4}C_3 = \frac{^{4}P_3}{3!} = \frac{4*3*2}{3*2*1} = 4$

   In *R*

   ```
   choose(4,3)
   [1] 4
   ```

Examples

1. If a box contains 75 good IC chips and 25 defective chips, and 12 chips are selected at random, find the probability that all chips are good.

$$E = \text{“All chips are good”}.$$

$$P(E) = \frac{\binom{75}{12}}{\binom{100}{12}}$$

To calculate $P(E)$ from *R* :

```
choose(75,12)/choose(100,12)
[1] 0.02486992
```

2. Choose a sample of ten from a class of 100 consisting of 60 females and 40 males.
   What is the probability of getting 10 females?

3. A box with fifteen integrated circuit chips contains five defectives. If a random sample of three chips is drawn, what is the probability that all three are defective?

4. In a party of five persons, compute the probability that at least two have the same birthday (month/day), assuming a 365 day year.

5. In a class of 100, what is the probability that at least 2 have the same birthday?