5 Rules of Probability

TYPES OF EVENTS

- **UNION OF TWO EVENTS**

  $E_1 \cup E_2$ denotes the outcome of E_1 **or** E_2

  Example: Pull a card from a deck.

  E_1 =Spade $\quad\quad E_2 =$ Ace

  $E_1 \bigcup E_2 \;\; = \;$ Ace or Spade

- **INTERSECTION OF TWO EVENTS**

  $E_1 \cap E_2$ denotes the outcome of E_1 **and** E_2

  Example: Pull a card from a deck.

  E_1 =Spade $\quad\quad E_2 =$ Ace

  $E_1 \cap E_2 \;\; = \;$ Ace and Spade

  $P(E_1 \cap E_2) \;\; = \;$ P(Ace and Spade)

- **MUTUALLY EXCLUSIVE EVENTS**

  E_1 and E_2 are mutually exclusive if they cannot occur together.

  Example: Pull a card from a deck.

  E_1 =Spade $\qquad E_2 =$ Heart

  $E_1 \cap E_2$ is impossible

  $E_1 \cap E_2 = \emptyset$

AXIOMS OF PROBABILITY

A probability function P is defined on subsets of the sample space **S** to satisfy the following axioms:

1. Non-Negative Probability:
$$P(E) \geq 0.$$

2. Mutually-Exclusive Events:
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
provided E_1 and E_2 are mutually exclusive.
i.e. $E_1 \cap E_2$ is empty.

3. The Universal Set:
$$P(S) = 1$$

Example:

Consider the following **if** statement in a program:
if B **then** s_1 **else** s_2
The random experiment consists of "observing" two successive executions of the **if** statement. The sample space consists of the four possible outcomes:

$$S = \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}$$

Assume that on the basis of strong experimental evidence the following probability assignment is justified:

$$P(s_1, s_1) = 0.34,\ P(s_1, s_2) = 0.26,\ P(s_2, s_1) = 0.26,\ P(s_2, s_2) = 0.14,$$

Calculate the probability of

1. of at least one execution of the statement s_1
2. that statement s_2 is executed first.

Solution:

1. Let $E =$ At least one execution of the statement s_1

$$E = \{(s_1, s_1),\ (s_1, s_2),\ (s_2, s_1)\}$$

$$P(E_1) = P(s_1, s_1) + P(s_1, s_2) + (s_2, s_1) = 0.86$$

2. Let $E =$ Statement s_2 is executed first.

$$E = \{(s_2, s_1),\ (s_2, s_2)\}$$

$$P(E) = P(s_2, s_1) + P(s_2, s_2) = 0.40$$

Properties of Probability

Theorem 1: Complementary Events

For each $E \subset S$:

$$P(\overline{E}) = 1 - P(E)$$

Proof:

$$S = E \cup \overline{E}$$

Now, E and $\overline{E}$ are mutually exclusive.
i.e.

$$E \cap \overline{E} \text{ is empty.}$$

Hence:

$$P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E}) \tag{Axiom2}$$

Also:

$$P(S) = 1 \tag{Axiom3}$$

i.e.

$$\begin{aligned} P(S) &= P(E) + P(\overline{E}) \\ \longrightarrow \quad 1 &= P(E) + P(\overline{E}) \end{aligned}$$

So:

$$P(\overline{E}) = 1 - P(E)$$

Properties of Probability

Theorem 2: The Impossible Event/The Empty Set

$$P(\emptyset) = 0 \text{ where } \emptyset \text{ is the empty set}$$

Proof:

$$S = S \cup \emptyset$$

Now: S and $\emptyset$ are mutually exclusive.
i.e.

$$S \cap \emptyset \text{ is empty.}$$

Hence:

$$P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$$

(Axiom2)

Also:

$$P(S) = 1$$

(Axiom3)

i.e.

$$1 = 1 + P(\emptyset)$$

i.e.

$$P(\emptyset) = 0.$$

Properties of Probability

Theorem 3:

If E_1 and E_2 are subsets of S such that $E_1 \subset E_2$, then

$$P(E_1) \leq P(E_2)$$

Proof:

$$E_2 = E_1 \cup (\overline{E}_1 \cap E_2)$$

Now, since E_1 and $\overline{E}_1 \cap E_2$ are mutually exclusive,

$$\begin{aligned} P(E_2) &= P(E_1) + P(\overline{E}_1 \cap E_2) \\ &\geq P(E_1) \end{aligned}$$

since $P(\overline{E}_1 \cap E_2) \geq 0$ from Axiom 1.

Properties of Probability

Theorem 4: Range of Probability

For each $E \subset S$

$$0 \leq P(E) \leq 1$$

Proof:
Since,

$$\emptyset \subset E \subset S$$

then from Theorem 3,

$$P(\emptyset) \leq P(E) \leq P(S)$$

$$0 \leq P(E) \leq 1$$

Theorem 5: The Addition Law of Probability

If E_1 and E_2 are subsets of S then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof:

$$E_1 \cup E_2 = E_1 \cup (E_2 \cap \overline{E}_1)$$

Now, since E_1 and $E_2 \cap \overline{E}_1$ are mutually exclusive,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2 \cap \overline{E}_1) \tag{1}$$

(Axiom2)

Now E_2 may be written as two mutually exclusive events as follows:

$$E_2 = (E_2 \cap E_1) \cup (E_2 \cap \overline{E}_1)$$

So

$$P(E_2) = P(E_2 \cap E_1) + P(E_2 \cap \overline{E}_1)$$

(Axiom2)

Thus:

$$P(E_2 \cap \overline{E}_1) = P(E_2) - P(E_2 \cap E_1) \tag{2}$$

Inserting (2) in (1), we get

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Example:
In a computer installation, 200 programs are written each week, 120 in C^{++} and 80 in Java.
60% of the programs written in C^{++} compile on the first run
80% of the Java programs compile on the first run.

What is the probability that a program chosen at random:

1. is written in C^{++} or compiles on first run?

2. is written in Java or does not compile?

3. either compiles or does not compile?