## 6 CONDITIONAL PROBABILITY

## Example:

In a computer installation, 200 programs are written each week, 120 in $C^{++}$and 80 in Java. $60 \%$ of the programs written in $C^{++}$compile on the first run and $80 \%$ of the Java programs compile on the first run.

|  | Compiles <br> on first run | Does not compile <br> on first run |  |
| :--- | :---: | :---: | :---: |
| $C^{++}$ | 72 | 48 | 120 |
| Java | 64 | 16 | 80 |
|  | 136 | 64 | 200 |

What is the probability that a program chosen at random:

1. compiles in the first run?
2. is written in $C^{++}$and compiles on first run?
3. compiles on the first run if we know it has been written in $C^{++}$?

## Conditional Probability

DEFN: The conditional probability of $\mathbf{B}$ given $\mathbf{A}$ is

$$
P(B \mid A)=\frac{P(A \cap B)}{P(A)}
$$

Terminology:

- Joint Probability: $\quad P(A \cap B)$
- Marginal Probability: $\quad P(A), \mathrm{P}(\mathrm{B})$
- Conditional Probability: $P(A \mid B)$ or $P(B \mid A)$

A rearrangement of the above definition yields the following:

## Multiplication Law of Probability:

Two events

$$
P(A \cap B)=\begin{aligned}
& P(A) P(B \mid A) \\
& P(B) P(A \mid B)
\end{aligned}
$$

More than two events:

$$
\begin{gathered}
P\left(E_{1} \cap E_{2} \cap E_{3} \cdots \cap E_{k}\right)= \\
P\left(E_{1}\right) P\left(E_{2} \mid E_{1}\right) P\left(E_{3} \mid E_{1} \cap E_{2}\right) \cdots P\left(E_{k} \mid E_{1} \cap E_{2} \cdots \cap E_{k-1}\right)
\end{gathered}
$$

## Examples:

1. Consider four computer firms $A, B, C, D$ bidding for a certain contract. A survey of past bidding success of these firms show the following probabilities of winning:

$$
P(A)=0.35, P(B)=0.15, P(C)=0.3, P(D)=0.2
$$

Before the decision is made to award the contract, firm B withdraws the bid. Find the new probabilities of winning the bid for A, $C$ and D.
2. Pull three cards from a deck without replacement. What is the probability that all are black?

## Independent Events

DEFN: Independent Events
$A$ and $B$ are said to be independent if

$$
P(A \mid B)=P(A)
$$

## Multiplication Law

## Two Independent Events

$$
P(A \cap B)=P(A) P(B)
$$

More than Two Independent Events

$$
P\left(E_{1} \cap E_{2} \cap E_{3} \cdots E_{k}\right)=P\left(E_{1}\right) P\left(E_{2}\right) \cdots P\left(E_{k}\right)
$$

Example: Draw 10 cards from a deck with replacement. What is the probability that all are black?

## Intel Chips

In October 1994, a flaw was discovered in the Pentium chip installed in many new personal computers. The chip produced an incorrect result when dividing two numbers. Intel, the manufacturer of the Pentium chip, initially announced that such an error would occur once in 9 billion divides, or "once in 27,000 years" for a typical user; consequently it did not immediately offer to replace the chip.
(a) For a division performed using the flawed chip, what is the probability that no error will occur?
(b) Consider two successive divisions performed using the flawed chip. What is the probability that neither result will be in error? (Assume that any one division has no impact on any other division.)
(c) Depending on the procedure, statistical software packages may perform an extremely large number of divisions to produce the required output. For heavy users of the software, one billion divisions over a short time frame is not unusual. Calculate the probability that 1 billion divisions performed using the flawed Pentium chip will result in no errors.
(d) Compute the probability that at least one error occurs in the 1 billion divisions.

Probability of an error in 1 divide

$$
P(E)=\frac{1}{9 \text { billion }}
$$

and the probability of no error in one divide

$$
P(\bar{E})=1-\frac{1}{9 \text { billion }}
$$

The probability that 1 billion divisions performed using the flawed Pentium chip will result in no errors is

$$
P(\bar{E})=\left(1-\frac{1}{9 \text { billion }}\right)^{1 \text { bill }}
$$

Probability of at least one error in one billion divides

$$
P(\text { at least one error })=1-\left(1-\frac{1}{9 \text { billion }}\right)^{1 \text { bill }}
$$

We use $R$ to calculate the value:
error <- 1/9000000000
noerror <- 1 - error
bill<-1000000000 \#number of divides
noerrorbill<- noerror**bill
noerrorbill
returns
[1] 0.8948393
atleastonebill <- 1-noerrorbill
atleastonebill
[1] 0.1051607

- over ten percent.


Two months after the flaw was discovered, Intel agreed to replace all Pentium chips free of charge.

## Law of Total Probability

If a sample space can be partitioned into $k$ mutually exclusive and exhaustive events:

$$
A_{1}, A_{2}, A_{3}, \cdots A_{k}
$$

i.e.

$$
S=A_{1} \cup A_{2} \cup A_{3} \cdots \cup A_{k}
$$

Then for any event $E$ :

$$
P(E)=P\left(A_{1}\right) P\left(E \mid A_{1}\right)+P\left(A_{2}\right)\left(P E \mid A_{2}\right) \cdots P\left(A_{k}\right) P\left(E \mid A_{K}\right)
$$

## Proof:

$$
\begin{aligned}
E & =E \cap S \\
& =E \cap\left(A_{1} \cup A_{2} \cup \cdots \cup A_{k}\right) \\
& =\left(E \cap A_{1}\right) \cup\left(E \cap A_{2}\right) \cup \cdots\left(E \cap A_{k}\right)
\end{aligned}
$$

Since these are mutually exclusive

$$
\begin{aligned}
P(E) & =P\left(E \cap A_{1}\right)+P\left(E \cap A_{2}\right)+\cdots P\left(E \cap A_{k}\right) \\
& =P\left(A_{1}\right) P\left(E \mid A_{1}\right)+P\left(A_{2}\right) P\left(E \mid A_{2}\right)+\cdots P\left(A_{k}\right) P\left(E \mid A_{k}\right)
\end{aligned}
$$

## Examples:

1. In a computer installation, $60 \%$ of programs are written in $C^{++}$ and $40 \%$ in Java. $60 \%$ of the programs written in $C^{++}$compile on the first run and $80 \%$ of the Java programs compile on the first run.
(a) What is the overall proportion of programs that compile on first run?
2. In a certain company

50\% of documents are written in WORD; 30\% in LATEX; 20\% in HTML.
From past experience it is know that :
$40 \%$ of the WORD documents exceed 10 pages
$20 \%$ of the LATEX documents exceed 10 pages
$20 \%$ of the HTML exceed 10 pages
(a) What is the overall proportion of documents containing more than 10 pages?

## Examples:

3. Enquiries to an on-line computer system arrive on 5 communication lines. The percentage of messages received through each line are:

| Line | 1 | 2 | 3 | 4 | 5 |
| :--- | ---: | ---: | ---: | ---: | ---: |
| \% received | 20 | 30 | 10 | 15 | 25 |

From past experience, it is known that the percentage of messages exceeding 100 characters on the different lines are:

| Line | 1 | 2 | 3 | 4 | 5 |
| :--- | ---: | ---: | ---: | ---: | ---: |
| $\%$ exceeding |  |  |  |  |  |
| 100 characters | 40 | 60 | 20 | 80 | 90 |

(a) Calculate the overall proportion of messages exceeding 100 characters.

Tree Diagram


## Examples:

4. A binary communication channel carries data as one of two sets of signals denoted by 0 and 1 . Owing to noise, a transmitted 0 is sometimes received as a 1 , and a transmitted 1 is sometimes received as a 0 . For a given channel, it can be assumed that a transmitted 0 is correctly received with probability 0.95 and a transmitted 1 is correctly received with probability 0.75 . Also, $70 \%$ of all messages are transmitted as a 0 . If a signal is sent, determine the probability that:
(a) a 1 was received;
(b) a 0 was received;
(c) an error occurred.
.95 (0)
Tree Diagram

25 (0)
. 3 (1)

