The Analytic Center Feasibility Pump

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Introduction

- The problem of finding a feasible solution of a generic mixed integer linear problem (MILP) is a NP-hard problem.

$$\begin{array}{rl} \min_{x} & c^T x \\ \text{s. to} & Ax = b \\ & x \geq 0 \\ & x_j \quad integer \quad \forall j \in \mathcal{I}, \end{array}$$

- Feasibility Pump (FP) (Fischetti, Glover, Lodi, 2005): A succesful heuristic for finding feasible solutions of Mixed Integer Linear Problems (MILPs).
  - Objective FP (Achterberg, Berthold, 2007): A slight modification of FP in order to improve the quality of the solutions.
  - Analytic Center FP (Baena, Castro, 2011): Interior Point Methods (IPMs) to Standard or Objective FP Heuristic.

Standard Feasibility Pump

- The FP heuristic starts by solving the linear programming (LP) relaxation of the MIP problem to generate a point x^* which is rounded to the nearest integer point $\tilde{x}$.
  - The point x^* satisfies the linear constraints, while $\tilde{x}$ satisfies the integrality constraints.
- FP heuristic finds the point x^* closest to $\tilde{x}$, by solving the following LP (using the L_1 Norm):

$$x^* = \arg\min\{\Delta(x, \tilde{x}) = \sum_{j \in \mathcal{I}} |x_j - \tilde{x}_j| : Ax = b, x \geq 0\},$$

- If $\Delta(x^*, \tilde{x}) = 0$ then x^* satisfies the linear and integrality constraints. If not, FP finds a new integer point $\tilde{x}$ from x^* by rounding.
- The pair of points $(\tilde{x}, x^*)$ are iteratively updated at each FP iteration with the aim of reducing the distance $\Delta(x^*, \tilde{x})$.

Objective Feasibility Pump

- Slight modification of the standard FP to find better solutions in terms of objective value.
- Consider a convex combination of objective function of MILP ($c^T x$) and $\Delta(x, \tilde{x})$:

$$\Delta_\alpha(x, \tilde{x}) := (1 - \alpha)\Delta(x, \tilde{x}) + \alpha \frac{||\Delta||}{||c||} c^T x, \quad \alpha \in [0, 1]$$

- Objective FP concentrates the search of a feasible solution on the region of high-quality points.
- α could be decreased at each iteration to put emphasis in feasibility. Standard FP is obtained with $\alpha = 0$.

Analytic Center

- The Analytic Center ($\bar{x}$) was computed by applying a primal-dual path following interior-point algorithm to the barrier problem of LP relaxation, after removing the objective function term (c=0):

$$\begin{aligned} \bar{x} = \min_{x} \quad & -\mu \textstyle\sum_{i=1}^{n} \ln x_i \\ \text{s. to} \quad & Ax = b \\ & x > 0, \end{aligned}$$

- AC depends on how the polytope $P = \{x \in \mathbb{R}^n \ : \ Ax = b, x \geq 0\}$ is represented.
    - Center of gravity is not affected by different formulations of the same polyhedron but is computationally more expensive.

Using AC in the feasibility pump heuristic

- AC ($\bar{x}$) can be used as a clever search through the following feasible segment:

$$x(\gamma) = \gamma\bar{x} + (1-\gamma)x^* \quad \gamma \in [0,1]$$

- Goal: To increase (in theory infinitely) the number of candidate feasible points to be rounded.
- In AC-FP, each $x(\gamma)$ is rounded to $\tilde{x}(\gamma)$. If feasible integer, STOP.
- If no feasible solution found at segment $\overline{\bar{x}x^*}$ two options to continue:
  - Using the point $\tilde{x}(0) = [x^*]$ (option $\gamma = 0$). Behaviour as standard/objective FP.
  - Using the point $\tilde{x}(\gamma)$ that minimizes $||\tilde{x}(\gamma) - x(\gamma)||_\infty$ (option L_∞). More chances to become feasible and integer. Better results in general.

Computational Results: Details

- AC-FP was implemented using the base code of the objective FP (C++).
- The base FP implementation was extended for computing the analytic center using three different interior points solvers (CPLEX, PCx and GLPK).
  - Cplex was not always able to provide the right analytic center because of aggressive reduced preprocessing (which cannot be deactivated).
  - GLPK discarded. PCx much more efficient.
- Applied to a subset of MIPLIB2003 instances.
- All runs were carried on a Dell PowerEdge 6950 server, four dual core AMD Opteron 8222 3.0 Ghz processors, 64 GB of RAM. Without use of parallelism capabilities.

Computational Results: Only binary variables

	AC-FP				objective FP			
Instance	niter	tFP(tAC)	stage	gap%	niter	tt	stage	gap%
10teams	179	26(0)*	3	10.59	**278**	**19**	**3**	**9.73**
a1c1s1	0	0(0)**	0	232	**351**	**8**	**2**	**97.45**
aflow30a	171	1(0)*	2	228.13	**41**	**0**	**1**	**103.28**
aflow40b	54	2(0)**	1	503.25	**21**	**1**	**1**	**99.32**
air04	186	1147(0)**	3	26.87	**45**	**181**	**1**	**3.73**
air05	190	148(0)**	3	35.73	**3**	**2**	**1**	**2.11**
cap6000	**0**	**1(0)****	**0**	**0.35**	31	0	1	0.38
danoint	99	4(0)*	1	15.50	**96**	**3**	**1**	**12.50**
disctom	**4**	**4(0)****	**1**	**0**	**3**	**3**	**1**	**0**
ds	**0**	**1(2)****	**0**	**5633.77**	446	9495	3	5633.77
fast0507	0	2(1)**	0	57.71	**8**	**51**	**1**	**5.71**
fiber	**15**	**0(0)****	**1**	**675.45**	41	0	1	1496.68
fixnet6	**18**	**0(0)***	**1**	**863.91**	67	0	1	936.77
glass4	**224**	**1(0)****	**3**	**316.67**	374	1	3	958.34
harp2	59	1(0)**	1	32.67	**138**	**3**	**1**	**17.90**
markshare1	**65**	**0(0)***	**1**	**30100**	65	0	1	36200
markshare2	**66**	**0(0)***	**1**	**46200**	65	0	1	48100
mas74	0	0(0)**	0	423649728.01	**109**	**0**	**1**	**40.10**
mas76	0	0(0)*	0	67000682.38	**106**	**1**	**1**	**15.59**
misc07	**219**	**2(0)****	**2**	**21.34**	188	1	1	31.31
mkc	13	1(0)*	1	50.79	**13**	**0**	**1**	**48.67**
mod011	23	3(1)*	1	31.30	**12**	**1**	**1**	**16.36**
modglob	**60**	**1(0)***	**1**	**5.16**	60	0	1	10.87
net12	**25**	**8(27)****	**1**	**57.21**	216	12	2	57.21
nsrand-ipx	**694**	**265(0)****	**3**	**296.56**	132	5	2	312.38

*: Compute AC with PCx Solver

**: Compute AC with Cplex Solver

Computational Results: Only binary variables

	AC-FP				objective FP			
Instance	niter	tFP(tAC)	stage	gap%	niter	tt	stage	gap%
nw04	2	9(8)*	1	9	**10**	**10**	**1**	**5.91**
opt1217	124	0(0)*	1	22.80	**40**	**0**	**1**	**0**
p2756	**279**	**7(0)****	**3**	**1542.85**	**377**	**2**	**3**	**1542.85**
pk1	57	0(0)*	1	625	**56**	**0**	**1**	**208.33**
pp08aCUTS	11	0(0)*	1	122.98	**10**	**0**	**1**	**13.74**
pp08a	15	0(0)*	1	115.63	**11**	**0**	**1**	**63.39**
protfold	**307**	**365(2)****	**3**	**37.81**	286	90	2	46.88
qiu	41	1(0)*	1	748.05	**9**	**0**	**1**	**219.34**
set1ch	0	0(0)**	0	296.92	**46**	**0**	**1**	**75.74**
seymour	0	0(0)**	0	38.92	**7**	**3**	**1**	**11.32**
sp97ar	63	57(4)*	1	75.87	**9**	**4**	**1**	**39.21**
swath	795	100(0)**	3	7324.22	**395**	**14**	**2**	**7575.56**
tr12-30	221	6(0)**	3	118.78	**25**	**1**	**1**	**25.68**
vpm2	27	0(0)**	1	67.8	**12**	**0**	**1**	**30.51**

*: Compute AC with PCx Solver

**: Compute AC with Cplex Solver

- AC-FP only obtain solutions with lower gap than FP in 14 of the 39 instances.

Computational Results: Binary and general integer variables

	AC-FP				objective FP			
Instance	niter	tFP(tAC)	stage	gap%	niter	tt	stage	gap%
arki001	871	43(0)*	3	1.96	**803**	**15**	**3**	**1.83**
atlanta-ip	**397**	**934(11)****	**3**	**70.32**	454	227	3	75.52
gesa2-o	**35**	**1(0)****	**2**	**26.59**	33	1	2	40.44
gesa2	3	1(0)*	2	49.23	**33**	**0**	**2**	**9.32**
manna81	**0**	**0(6)***	**0**	**1.64**	52	2	2	1.70
msc98-ip	**33**	**16(949)***	**1**	**52.20**	61	26	1	53.75
mzzv11	567	435(116)*	3	25.12	**540**	**127**	**3**	**17.59**
mzzv42z	27	12(15)**	1	30.90	**25**	**49**	**1**	**29.39**
noswot	33	0(0)**	2	23.81	**13**	**1**	**2**	**0**
roll3000	**175**	**11(1)****	**2**	**43.57**	793	17	3	180.12
rout	**74**	**1(0)****	**1**	**24.08**	117	0	1	53.31
timtab1	**169**	**1(0)***	**2**	**41.35**	216	1	2	83.13
timtab2	972	6(0)**	3	91.96	1222	2	2	80.75

*: Compute AC with PCx Solver

**: Compute AC with Cplex Solver

- AC-FP obtain results with lower gap than FP in 8 of the 13 instances.

The analytic center feasibility method (ACFM)

- Analytic Center Feasibility Method (Naoum-Sawaya, Elhedli,2011): Use a cutting plane approach where the AC is computed repeteadly.
- ACFM computes a weighted analytic center (updating upper bounds or adding additional constraints).
- ACFM considers two line segments: $\overline{\bar{x}\ x^*_{\text{min}}}$ and $\overline{\bar{x}\ x^*_{\text{max}}}$.
  - x^*_{min} and x^*_{max} are the minimizer and maximizer, respectively, of the objective function of the LP relaxation.
  - Candidate integer solutions are found by rounding the solutions on $\overline{\bar{x}\ x^*_{\text{min}}}$ and $\overline{\bar{x}\ x^*_{\text{max}}}$ to the nearest integer.
- If integer feasible solution found, update upper bound, recompute weighted AC and continue the search.
- Else: Additional constraints added to shift the AC to a new position. Greater weight is given to constraints violated by the rounded points. Continue the search.

Differences

- ACFM computes one analytic center (for a modified polyhedron) at each iteration. So, ACFM is computationally more inefficient since the computation of AC can be expensive.
- ACFM and AC-FP are completely different approaches:
  - AC-FP is an extension of FP.
  - ACFM is based on computing analytic centers of modified polyhedrons obtained by adding cutting planes to P.
- In AC-FP the analytic center is the same for all the iterations and x^* is different at each iteration. In ACFM the AC is different at each iteration and x^*_{min}, x^*_{max} is the same for all the iterations.
- Initially, AC-FP also considered two segments: the current and the farthest feasible point from $\bar{x}$ in direction $\bar{x} - x^*$ (name it x^*_f).
  - Computational benefit of using x^*_f instead of x^*_{max} is that the solution of an extra LP problem is avoided.
  - In practice, using the second segment $\overline{\bar{x}x^*_f}$ was not useful.

Computational results: Comparison

	AC-FP				ACFM		
Instance	niter	tFP(tAC)	stage	gap%	niter	tt(tAC)	gap%
mas74	0	0(0)**	0	423649728.01	**7**	**8.89(8.26)**	**434.75**
mas76	0	0(0)*	0	67000682.38	**1**	**2.55(2.1)**	**12.18**
misc07	**219**	**2(0)****	**2**	**21.34**	13	9.28(8.71)	70.64
noswot	33	0(0)**	2	23.81	**3**	**2.51(2.11)**	**9.76**
pk1	57	0(0)*	1	625	**1**	**0.75(0.72)**	**163.55**
pp08aCUTS	11	0(0)*	1	122.98	**1**	**2.81(2.25)**	**15.07**
pp08a	15	0(0)*	1	115.63	**1**	**2.07(1.5)**	**23.11**
rout	74	1(0)**	1	24.08	**4**	**101.95(100.58)**	**3.18**
vpm2	27	0(0)**	1	67.8	**6**	**28.43(27.31)**	**12.73**

*: Compute AC with PCx Solver
**: Compute AC with Cplex Solver

- ACFM seems to provide better points, but it is computationally much more expensive (it computes one AC per iteration, AC-FP only computes one AC).
- ACFM was only tested on nine of the smaller MIPLIB2003 instances. AC-FP was tested on 54 (some of them much larger) instances.

Conclusion

- Feasibility Pump (Fischetti, Glover, Lodi): A succesful heuristic for finding feasible solutions of Mixed Integer Linear Problems (MILPs).
- Use the Analytic Center of the polyhedron to round points in a segment of LP-feasible points: points in this segment are "interior", so more chances to get a rounded MILP-feasible point.
- For general 0-1 problems, no improvement observed over standard/objective feasibility pump.
- For problems with both general integer and binary variables, and for some particular binary problems, the AC-FP may result in more efficient and lower gap solutions.
- An example of how Interior Point methods and Simplex may collaborate to get a MILP feasible solution.
- Work published in **D. Baena, J. Castro, Using the analytic center in the feasibility pump, Operations Research Letters, 39 (2011) 310-317**

Thanks