A small sample comparison of maximum likelihood, moments and L-moments methods for the asymmetric exponential power distribution

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Abstract

This article considers three methods of estimation, namely maximum likelihood, moments and L-moments, when data come from an asymmetric exponential power distribution. This is a four parameters very flexible parametric family exhibiting variety of tail and shape behaviour. The analytical expression of the first four L-moments of these distributions are derived, what allows the use of L-moments estimators. A simulation study compares the three estimation methods in small samples.

Key words: Asymmetric distribution; Heavy tails distribution; Mean Square Error; Non-parametric mode estimation; Numerical optimization; Simulation output data analysis.

Running headline: L-moments for As.Exp.Power distribution.

1 Introduction

Hosking, Wallis, and Wood (1985) and Hosking and Wallis (1987) applied L-moments estimation method to extreme value distribution. They found that it performs better than method of moments and that both methods do well in small samples compared to maximum likelihood estimation. However, these studies exclusively refer to meteorological data. Our objective is to enlarge upon these previous studies by applying these methods to a general class of models with application in other fields. We will investigate whether similar conclusions can be reached.

For this purpose, we consider an asymmetric exponential power distribution, introduced and discussed by Ayebo and Kozubowski (2003). This family of distributions was obtained by the authors by incorporating inverse scale factors into the negative and positive orthants in generalized error distribution. It includes skewed Normal and skewed Laplace, studied respectively by Mudholkar and Hutson (2000) and Kotz, Kozubowski, and Podgórski (2001), quite useful for modeling in finance, economics and the sciences. Mudholkar and Hutson (2000) call their proposal *epsilon-skew-normal* distribution to differentiate it from the skew-normal distribution proposed by Azzalini (1985) (see also Azzalini 2005). The relationship between both definitions is analyzed in Section 3.

The choice of this flexible four parameters model lies in the fact that besides exhibiting variety of tail and shape behaviour, all three methods of estimation are applicable. A heavier tailed choice would have ruled out the method of moments. Method of moments and maximum likelihood are well-known to all statisticians whereas L-moments method (related to L-statistics) has appeared mainly in meteorological literature.

It is standard practice to summarise the observed data by moments and fit a probability density function to data set by method of moments, indeed it was the only method used to fit a mixture to a data set before the advent of EM algorithm. It is known to be markedly less accurate than maximum likelihood. Furthermore the information conveyed by third and higher order moments about the shape of distribution are often difficult to assess, particularly in small sample, where the numerical values of sample moments can be very different from those of probability density function from which sample is drawn (for details see Kirby 1974).

The maximum likelihood method for estimating the parameter or fitting the probability density function to a data set is universally used including the mixture facilitated by the introduction of EM algorithm. Its acclaimed superiority resides in its established asymptotic propertiesi. In practice however, one has finite sample and asymptotic theory is not the reliable guide to finite sample performances (see Hannan 1987). Indeed often it gives worse results than suggested by asymptotic ones and in some cases yields parameter and quantile estimators which are less efficient than other methods.

The L-moments method being quite recent, we briefly describe it in the next section. Then in Section 3 we derive first four L-moments of asymmetric exponential power distributions, the expressions are quite complicated but they do simplify considerably in the symmetric case. In Section 4 we give some simulation results on three methods of estimation and summarize the results of simulation study. Finally the last section presents some concluding remarks.

2 L-moments and method of L-moments

The L-moments appeared without name for the first time in quantile expansion of Sillitto (1969). Hosking (1986) in his research report coined the name L-moments. Hosking (1990) unified scattered results of various authors and further added new results.

The L-moments and ordinary moments are special cases of probability weighted moments introduced by Greenwood, Landwehr, Matala, and Wallis (1979) as

$$M_{p,r,s} = \beta(r+1, s+1)E[X^p_{(r+1,r+s+1)}]$$

which exists for all $r, s \geq 0$ if and only if $E|X^p|$ is finite, where $X_{(r+1,r+s+1)}$ is $(r+1)-$ th order statistic in sample of $r+s+1$ size. Obviously $M_{p,0,0}$ are ordinary moments. Of special interest to us in the present context are

$$\{M_{1,r,0} = \beta_r = \frac{1}{r+1}E[X_{(r+1,r+1)}], r = 0, 1, \ldots\}$$

which uniquely characterise the distribution requiring only the existence of mean (see Chan 1967). The L-moments are linear function of expected order statistics and are defined as

$$\lambda_{r+1} = (r+1)^{-1}\sum_{k=0}^{r}(-1)^k\binom{r}{k}EX_{(r+1-k,r+1)} = \sum_{k=0}^{r} p_{r,k}\beta_r, r = 0, 1, \ldots,$$

$$p_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}.$$

Explicitly

$$\lambda_1 = \beta_0, \ \lambda_2 = 2\beta_1 - \beta_0, \ \lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0, \ \lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0.$$

Moreover these first four L-moments admit a more easily understandable expression:

$$\lambda_1 = \beta_0, \ \lambda_2 = \frac{1}{2} E(X_{(2,2)} - X_{(1,2)}), \ \lambda_3 = \frac{1}{3} E(X_{(3,3)} - 2X_{(2,3)} + X_{(1,3)}),$$

$$\lambda_4 = \frac{1}{4} E(X_{(4,4)} - 3X_{(3,4)} + 3X_{(3,4)} - X_{(1,4)}).$$

It follows that λ_1, λ_2, λ_3/λ_2 and λ_4/λ_2 may be regarded as a measures of location, scale, skewness and kurtosis respectively (see Section 2.3 in Hosking 1990 for more details).

The sample L-moments are defined as

$$l_{r+1} = \sum_{k=0}^{r} p_{r,k} b_k, r = 0, 1, \ldots$$

where

$$b_r = n^{-1} \sum_{k=r+1}^{n} \binom{k-1}{r} \binom{n-1}{r}^{-1} x_{(k,n)}, \ r = 0, 1, \ldots, n-1,$$

and $x_{(k,n)}$ is the k-th order statistic. One can equally represent the l_r in terms of U statistics, i.e., the average over all sub-samples of size $r < n$. The method of L-moments consists in equating the sample L-moments to L-moments of distribution and solving for the parameters. The resulting estimators are consistent and asymptotically normal (for details see Hosking 1990).

The L-method is particularly handy for the models having quantile function explicitly expressible in terms of distribution function, specifically for the Tukey's lambda distribution both methods of moments and maximum likelihood are not straightforward to apply, compared to L-method. Furthermore for heavy tailed distributions with only finite mean, this is a viable alternative to maximum likelihood (see Mudholkar and Hutson 1998 for a class of estimators analogous to L-moments that always exist). The L-moments being linear functions of order statistic, they are subject to less sampling variability, robust to outliers and the asymptotic results are reliable guide even for small samples.

3 L-moments of asymmetric exponential power distribution

The asymmetric exponential power distribution has the following density function:

$$f(x) = \frac{\alpha\kappa}{\sigma(1+\kappa^2)\Gamma(1/\alpha)} \exp\left\{-\left(\kappa^{\mathrm{sgn}(x-\theta)}\left(\frac{|x-\theta|}{\sigma}\right)\right)^{\alpha}\right\}, -\infty < x < \infty, \qquad (1)$$

where $\mathrm{sgn}(u)$ is the sign of u. Parameters θ and $\sigma > 0$ correspond to location and scale respectively, whereas $\kappa > 0$ and $\alpha > 0$ deal with skewness and shape of distribution. To have some idea of variety of tail and shape behaviour exhibited by the above model, we give its graph for some selected value of the parameters in Figure 1.

Ayebo and Kozubowski (2003) follow a general procedure described in Fernández and Steel (1998) that allows to introduce a skewed version f of a given symmetric about 0 density function f_0:

$$f_k(x) = 2\frac{k}{1+k^2} f_0(x\kappa^{\mathrm{sgn}(x)}),\ k > 0.$$

A different mechanism appears in Azzalini (2005), where the skewed version of f_0 is

$$f_{\mathrm{Az}}(x) = 2f_0(x)G(w(x)),$$

where G is the distribution function of an absolutely continuous random variable symmetric about 0, and w is an odd function. The following result (that can be easily verified) establishes that under certain conditions the first asymmetrization mechanism is a particular case of the second one.

Proposition 1 *If f_0 verifies that $\lim_{x\to\infty} f_0(\delta x)/f_0(x) = 0$ for all $\delta > 1$, then for all x $f_k(x) = f^*_{\mathrm{Az}}(x) = 2f^*_0(x)G(w(x))$, where*

$$f^*_0(x) = \frac{k}{1+k^2}(f_0(xk) + f_0(x/k)),\ G(x) = \frac{f_0(xk^{\mathrm{sgn}(x)})}{f_0(xk) + f_0(x/k)},\ w(x) = \mathrm{sgn}(1-k)x.$$

The condition on the tail behaviour of f_0 is needed to show that G is indeed a distribution function. It is fulfilled when f_0 is the *exponential power distribution* (in this case $f_0(x) \propto \exp(-|x/\sigma|^{\alpha})$) that is the symmetric density used in equation (1) to define $f(x)$.

The choice of model (1) on the one hand enlarges the previous studies on L-moments dealing exclusively with extreme values distribution and further allows to verify the claim by Hosking, Wallis, and Wood (1985) that for models with at least three parameters, the L-methods fairs better than the other two.

To compute the first four L-moments, we need to find

$$\beta_r = \int xF(x)^r f(x)dx, r = 0, 1, 2, 3,$$

where F is

$$F(y) = \begin{cases} \frac{\kappa^2}{(1+\kappa^2)}\Gamma(1/\alpha, [\frac{(\theta-y)}{\sigma\kappa}]^\alpha), & y < \theta \\ 1 - \frac{1}{(1+\kappa^2)\Gamma(1/\alpha)}\Gamma(1/\alpha, [\frac{\kappa(y-\theta)}{\sigma}]^\alpha), & y \geq \theta, \end{cases}$$

the distribution function of the asymmetric exponential power random variable, f is its probability density function and $\Gamma(a, x)$ is the normalized incomplete Gamma function.

It is not hard to verify the following statements. First, if X has asymmetric exponential power distribution with parameters $\theta = (0, \sigma, \kappa, \alpha)$, then $-X$ has also exponential power distribution with parameters $\theta = (0, \sigma, 1/\kappa, \alpha)$. Second, if $\theta = 0$, $\beta_r(-X)$ can be obtained from $\beta_r(X)$ by replacing κ by $1/\kappa$, furthermore

$$\beta_r(-X) = -\int x([1 - F(x)]^r f(x)dx = -\alpha_r(X).$$

Consequently it can be easily verified (see Hosking 1990) that $\lambda_3(X) = -\lambda_3(-X)$ whereas

$$\lambda_{2r}(X) = \lambda_{2r}(-X),\ r = 1, 2.$$

This will be used as double check for the computation of L-moments.

Obviously

$$\lambda_1 = \beta_0 = EX = \theta + \frac{\sigma(1/\kappa - \kappa)\Gamma(2/\alpha)}{\Gamma(1/\alpha)}.$$

Note that

$$\lambda_r(X) = |b|\lambda_r(Y),\ X = a + bY,$$

consequently it is sufficient to find $\lambda_r/\sigma, r > 1$, from standardized exponential power distribution. By straightforward computation with $\theta = 0, \sigma = 1$, we find

$$\beta_1 = \int xF(x)f(x)dx = \frac{\kappa^3(1/\kappa^2 - \kappa^2)\Gamma(2/\alpha)}{(1+\kappa^2)^2\Gamma(1/\alpha)} + \frac{\kappa^2(\kappa^3 + 1/\kappa^3)\Gamma(2/\alpha)I_{1/2}(1/\alpha, 2/\alpha)}{(1+\kappa^2)^2\Gamma(1/\alpha)},$$

where $I_{1/2}(1/\alpha, 2/\alpha)$ is normalized incomplete beta function. Hence

$$\lambda_2 = -\frac{\sigma\kappa(1/\kappa - \kappa)^2\Gamma(2/\alpha)}{(1+\kappa^2)\Gamma(1/\alpha)} + 2\frac{\sigma\kappa^2(1/\kappa^3 + \kappa^3)\Gamma(2/\alpha)I_{1/2}(1/\alpha, 2/\alpha)}{(1+\kappa^2)^2\Gamma(1/\alpha)}.$$

Next

$$\beta_2 = \frac{\kappa^5(1/\kappa^2-\kappa^2)\Gamma(2/\alpha)}{(1+\kappa^2)^3\Gamma(1/\alpha)} + 2\frac{\kappa^4(\kappa^3+1/\kappa^3)\Gamma(2/\alpha)I_{1/2}(1/\alpha,2/\alpha)}{(1+\kappa^2)^3\Gamma(1/\alpha)}$$
$$+\frac{\kappa^3(1/\kappa^4-\kappa^4)\Gamma(2/\alpha)\Delta}{(1+\kappa^2)^3\Gamma(1/\alpha)},$$

where

$$\Delta = \frac{1}{\beta(1/\alpha,2/\alpha)}\int_0^{1/2} t^{(1/\alpha-1)}(1-t)^{(2/\alpha-1)}I_{(1-t)/(2-t)}(1/\alpha,3/\alpha)dt.$$

Consequently

$$\lambda_3 = \frac{\sigma(1/\kappa-\kappa)(\kappa^4-4\kappa^2+1)\Gamma(2/\alpha)}{(1+\kappa^2)^2\Gamma(1/\alpha)}$$
$$-\frac{6\sigma\kappa^3(1/\kappa-\kappa)(1/\kappa^3+\kappa^3)\Gamma(2/\alpha)I_{1/2}(1/\alpha,2/\alpha)}{(1+\kappa^2)^3\Gamma(1/\alpha)} + \frac{6\sigma(1+\kappa^4)(1/\kappa-\kappa)\Gamma(2/\alpha)\Delta}{(1+\kappa^2)^2\Gamma(1/\alpha)}.$$

Finally

$$\beta_3 = \frac{\kappa^7(1/\kappa^2-\kappa^2)\Gamma(2/\alpha)}{(1+\kappa^2)^4\Gamma(1/\alpha)} + \frac{3\kappa^6(1/\kappa^3+\kappa^3)\Gamma(2/\alpha)I_{1/2}(1/\alpha,2/\alpha)}{(1+\kappa^2)^4\Gamma(1/\alpha)}$$
$$+\frac{3\kappa^5(1/\kappa^4-\kappa^4)\Gamma(2/\alpha)\Delta}{(1+\kappa^2)^4\Gamma(1/\alpha)} + \frac{\kappa^4(\kappa^5+1/\kappa^5)\Gamma(2/\alpha)\Delta_1}{(1+\kappa^2)^4\Gamma(1/\alpha)},$$

where

$$\Delta_1 = \frac{\int_0^{1/2}\int_0^{\frac{1-y}{2-y}} y^{(1/\alpha-1)}(1-y)^{(2/\alpha-1)}z^{(1/\alpha-1)}(1-z)^{(3/\alpha-1)}I_{\frac{(1-z)(1-y)}{1+(1-z)(1-y)}}(1/\alpha,4/\alpha)dzdy}{\beta(1/\alpha,2/\alpha)\beta(1/\alpha,3/\alpha)}$$

and

$$\lambda_4 = -\frac{\sigma\kappa(1/\kappa-\kappa)^2(\kappa^4-8\kappa^2+1)\Gamma(2/\alpha)}{(1+\kappa^2)^3\Gamma(1/\alpha)}$$
$$+12\frac{\sigma\kappa^2(\kappa^3+1/\kappa^3)(\kappa^4-3\kappa^2+1)\Gamma(2/\alpha)I_{1/2}(1/\alpha,2/\alpha)}{(1+\kappa^2)^4\Gamma(1/\alpha)}$$
$$-30\frac{\sigma\kappa^3(1/\kappa-\kappa)^2(1/\kappa^2+\kappa^2)\Gamma(2/\alpha)\Delta}{(1+\kappa^2)^3\Gamma(1/\alpha)} + 20\frac{\sigma\kappa^4(1/\kappa^5+\kappa^5)\Gamma(2/\alpha)\Delta_1}{(1+\kappa^2)^4\Gamma(1/\alpha)}.$$

For symmetric exponential power distribution, i.e., $\kappa = 1$, we have $\lambda_1 = \theta, \lambda_3 = 0$ and also the expressions for λ_2, λ_4 considerably simplify.

4 Simulation study

We have developed a simulation study in order to gain an insight into the performance on small samples of the three estimation methods: maximum likelihood estimation (MLE), the moments method (MOM), and the L-moments method (LMO). Two main scenarios are examined: in the first one the location parameter θ is taken as known (results in Table 1) whereas in the second case all four parameters are assumed to be unknown (results are in Table 2).

The log-likelihood function is

$$\log L(\alpha,\kappa,\sigma,\theta) = n\left(\log\frac{\alpha}{\Gamma(1/\alpha)} + \log\frac{\kappa}{1+\kappa^2} - \log\sigma - \frac{\kappa^\alpha}{\sigma^\alpha}\bar{x}_\alpha^+ - \frac{1}{\kappa^\alpha\sigma^\alpha}\bar{x}_\alpha^-\right),$$

where

$$\bar{x}_\alpha^+ = \frac{1}{n}\sum_{i=1}^{n}([x_i-\theta]^+)^\alpha, \bar{x}_\alpha^- = \frac{1}{n}\sum_{i=1}^{n}([x_i-\theta]^-)^\alpha,$$

and $[x]^+ = \max\{x,0\}$, $[x]^- = \max\{-x,0\}$.

The formulas for computing the MLE of κ and σ (depending on α and θ) are taken from Ayebo and Kozubowski (2003):

$$\hat{\kappa} = \hat{\kappa}(\alpha,\theta) = \left[\frac{\bar{x}_\alpha^-}{\bar{x}_\alpha^+}\right]^{\frac{1}{2(\alpha+1)}}, \tag{2}$$

$$\hat{\sigma} = \hat{\sigma}(\alpha,\theta) = \left[\alpha(\bar{x}_\alpha^+\bar{x}_\alpha^-)^{\frac{\alpha}{2(\alpha+1)}}\left([\bar{x}_\alpha^+]^{\frac{1}{\alpha+1}} + [\bar{x}_\alpha^-]^{\frac{1}{\alpha+1}}\right)\right]^{\frac{1}{\alpha}}.$$

The function of α

$$\log L(\alpha, \hat{\kappa}(\alpha,\theta), \hat{\sigma}(\alpha,\theta), \theta)$$

has to be numerically maximized to determine the MLE of α. The value θ is consider to be known here.

When location parameter has to be estimated (second scenario) we follows the indication of Ayebo and Kozubowski (2003) and use a non-parametric mode estimator: the half-range mode estimation method (Bickel 2002). Let $\hat{\theta}$ be this estimator of θ. θ is replaced by $\hat{\theta}$ in the expression of $\bar{x}_\alpha^+$ and $\bar{x}_\alpha^-$ to obtain the corresponding log likelihood function $\log L(\alpha, \hat{\kappa}(\alpha,\hat{\theta}), \hat{\sigma}(\alpha,\hat{\theta}), \hat{\theta})$ that is maximized in α. An alternative method should be to estimate α and θ simultaneously by MLE, maximizing $\log L(\alpha, \hat{\kappa}(\alpha,\theta), \hat{\sigma}(\alpha,\theta), \theta)$ in both parameters. A numerical optimization in two variables should be conducted, and it would however be hard to distinguish between numerical and statistical performance of the estimation method. So we decide not to explore this way and follow Ayebo and Kozubowski (2003) advice.

For the moments method estimation we equate the empirical mean, variance and asymmetry coefficient to their theoretical counterpart and solve numerically the nonlinear system. We use the expressions given in Ayebo and Kozubowski (2003):

$$\mu = E(X) = \theta + \sigma \left(\frac{1}{\kappa} - \kappa \right) \frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)},$$

$$\sigma^2 = Var(X) = \sigma^2 \frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)} \frac{1+\kappa^6}{\kappa^2(1+\kappa^2)} - \sigma^2 \frac{\Gamma^2(2/\alpha)}{\Gamma^2(1/\alpha)} \frac{(1-\kappa^2)^2}{\kappa^2},$$

$$\gamma = E\left[\left(\frac{X-\mu}{\sigma} \right)^3 \right] =$$

$$\frac{(1-\kappa^8)\Gamma^2(\frac{1}{\alpha})\Gamma(\frac{4}{\alpha}) - 3(1-\kappa^2)(1+\kappa^6)\Gamma(\frac{1}{\alpha})\Gamma(\frac{2}{\alpha})\Gamma(\frac{3}{\alpha}) + 2(1-\kappa^2)^3(1+\kappa^2)\Gamma^3(\frac{2}{\alpha})}{(1+\kappa^2)\left(\Gamma(1/\alpha)\Gamma(3/\alpha)\frac{1+\kappa^6}{1+\kappa^2} - \Gamma^2(2/\alpha)(1-\kappa^2)^2\right)^{3/2}}.$$

Observe that only three theoretical moments are needed because, when it is necessary, parameter θ is estimated by the half-range mode method.

Estimation by the method of L-moments follows a similar mechanics: we look for the combinations of parameters α, κ, and σ solving the system of equations

$$\lambda_r = l_r, \; r = 1, 2, 3,$$

where λ_r are the theoretical L-moments (depending on parameters as it is stated in Section 3) and l_r are the empirical L-moments (computed as indicated in Section 2).

We have simulated samples of three different sizes ($n = 10, 20, 50$). Only a value for the location parameter ($\theta = 0$) and for the scale parameter ($\sigma = 1$) are used because the use of different location and/or scale values has no effect on the results. Four values for α (0.5, 1, 2, 4) and two for κ (1/2, 1) are considered. In this way we contemplated the double exponential case ($\alpha = 1$, $\kappa = 1$), the normal case ($\alpha = 2$, $\kappa = 1$), a more concentrated symmetric case ($\alpha = 4$, $\kappa = 1$), a very heavy tail symmetric distribution ($\alpha = 0.5$, $\kappa = 1$), and their right asymmetric versions ($\kappa < 1$). The case κ is essentially equivalent to the case $1/\kappa$ (only asymmetry direction changes), so we have only considered $\kappa \leq 1$. The number of simulated samples for each combination was $S = 5000$. Figure 1 shows the theoretical density functions for the simulated data.

(Figure 1 about here)

All the computations have been done in R (R Development Core Team 2005). In the MLE of α the one dimensional optimization were carried out with the R-function `optimize`, that uses a combination of golden section search and successive parabolic interpolation. For moments and L-moments estimation, the sum of squares of the differences between empirical and theoretical moments (or L-moments), as a functions of the unkown parameters, have been numerically minimized. The parameter combination where the minimum is achieved is considered as moments (or L-moments) estimator of the unknown parameters. The multivariate optimization R-function `optim` was used to minimize these sum of squares. A quasi-Newton method which allows box constraints (each variable can be given a lower and/or upper bound) has been chosen. For numerical stability reasons the objective functions were actually the logarithm of 1 plus the sum of squares, expressed as function of the logs of the parameters α, κ, and σ. All the numerical optimizations have been done restricting α to the interval $[.25,\ 6]$, κ to $[0.1,\ 1.2]$, and σ to $[0.5,\ 2]$.

Figure 2 shows the contour plots for the objective functions optimized to obtain the three type of estimators. The sample used to compute these functions has size $n = 20$ and was generated with $\alpha = 2$, $\kappa = 1$, $\sigma = 1$, and $\theta = 0$. The graphics in the left column use parameters as optimization variables, and logs of parameters are used in the right column. It can be seen that the contour level sets are more rounded when parameters are taken in logs. This fact helps in the numerical optimization process.

(Figure 2 about here)

Tables 1 and 2 show the results of the simulation. The first one corresponds to the case where θ is assumed to be known, and the second one to scenario with unknown θ. For each combination of n, α and κ, for each estimation method, and for each estimated parameter two numbers are displayed: the average over the $S = 5000$ simulations, and the squared root of the mean squared error (in brackets and in italic). The figures corresponding to $\hat{\theta}$ in Table 2 show the performance of the nonparametric estimation of θ.

(Table 1 about here)

(Table 2 about here)

We first briefly examine the estimation bias and then we deal with MSE in more detail. All three methods provide estimates with low bias for σ and large for α and for κ, this one specifically for asymmetric distributions. On the basis of absolute value of bias estimates, we see that L-method fairs better than others for shape parameter α whereas for estimating σ and κ, MLE and moments method share the lead. Obviously size of estimate bias reduces with increase of n.

A first attempt to analyze the squared root of MSE data could be to fit multifactorial ANOVA models. Six different models could be fitted, corresponding to the three estimated parameters (α, σ, κ) and the two scenarios (known or estimated θ). Each ANOVA model would include the four factors considered in the simulation (see factors and levels listed bellow) and perhaps second order interactions. Nevertheless our simulation data do not verify the required hypothesis fot fitting such multifactorial ANOVA models: in fact we are dealing with a balanced design with only three true factors (n, α and κ) and the response variable is three dimensional, because for each combination of those factors we observe the three values of square root of MSE (computed from the same set of $S = 5000$ simulated samples) corresponding to the three types of estimator we are comparing. So the specific estimator in use is not a factor design. Given that strictly speaking it is not possible to fit univariate multifactorial ANOVA models to the squared root of MSE data, we opted for doing a descriptive analysis of the data, according to the following factorial structure:

Factor	**Levels**	**Values**
`Estimator`	3	LMO; MLE; MOM
n	3	10; 20; 50
α	4	0,5; 1,0; 2,0; 4,0
κ	2	0,5; 1,0

Figure 3 shows the main effects plots corresponding to the levels of the above four factors. Their six panels are a graphical summary of the squared root of MSE values for Tables 1 (known θ, left column panels) and 2 (estimated θ, right column panels). Each point represents the mean of all the square root of MSE values corresponding to a combination of a factor value and an estimated parameter. The interaction between factors are well summarized by the interaction plots shown in Figures 4, 5, and 6. Here each point is the mean of all the square root of MSE values corresponding to a combination of values for two factors and an estimated parameter.

(Figure 3 about here)

(Figure 4 about here)

(Figure 5 about here)

(Figure 6 about here)

The following conclusions are derived from the simulation results and the main effects and interaction plots. We start with the estimation of α in the known θ case (see the first row first column panel in Figure 3 and the top panel in Figure 4). L-moments method is recomended for the estimation of α, mainly when n is small (10 or 20), and the true density has high tails (α small) or is asymmetric ($\kappa = 0.5$). When n grows, the MSE of α estimators decreases, and this happens much more quickly for MLE than for moments or L-moments estimators. It can be seen that the MSE of α estimators increases when the true α increases. This is because when the target parameter α is bigger the variability of the estimation is also bigger. Nevertheless, observe that the MSE corresponding to $\alpha = 4$ is lower than that corresponding to $\alpha = 2$: the more regular shape of the density function when $\alpha = 4$ compensates the increment of MSE due to the increment in α. The case corresponding to the estimation of α and unknown θ (estimated by mode estimation) is very similar to the previous one (see the first row second column panel in Figure 3 and the bottom panel in Figure 4). The precision in the estimation of α is similar for known or estimated θ, as the main effects plots show.

Let us go now to the estimation of σ in the known θ case (see the second row first column panel in Figure 3 and the top panel in Figure 5). In this case the moments estimator is the recommended one, due mainly to cases $\alpha < 2$ (heavy tail), $\kappa = 0.5$ (asymmetry) and $n = 10$. In other cases the MLE is the best one. The L-moments estimator is not recommended for the estimation of σ. The asymmetric case ($\kappa = .5$) brings better estimations of σ than the symmetric one ($\kappa = 1$). When θ is unknown (see the second row second column panel in Figure 3 and the bottom panel in Figure 5) the MLE is the recommended one. In this situation is the value of α what mainly determines the performance of the estimators (MSE values decrease with α). Another noteworthy difference with the known θ case is that now the quality of the estimations does not depend on κ (MSE values are practically constant in κ). The main effects plots show that σ is much better estimated when θ is known (see the ordinate scale of the second row of panels in Figure 3).

Finally we deal with the estimation of parameter κ, starting with the known θ case (see the third row first column panel in Figure 3 and the top panel in Figure 6). Here the recommended estimator is the one based in L-moments. It is comparable to MLE in the asymmetric case ($\kappa = 0.5$) and clearly better in the symmetric case ($\kappa = 1$). A general fact is that MSE in the estimation of κ is bigger for $\kappa = 1$ than for $\kappa = .5$. This happens again (as in estimation of α) because usually the bigger is the parameter the bigger is the estimators variability. These differences are more remarkable for MLE than for the other two estimation methods. The corresponding case when θ is unknown (see the third row second column panel in Figure 3 and the bottom panel in Figure 6) presents a notable difference with respect to the previous situation: now the MSE increases in α and decreases in κ, while the opposite happens in the known θ case. Now MLE and L-moments estimator perform very similarly and both beat the moments estimator. The main effects plots show that σ is much better estimated when θ is known (see the ordinate scale of the third row of panels in Figure 3).

As a summary we can say that L-moments method performs well when estimating α or κ. That is more clear for $n = 10$ and $n = 20$, specially for heavy tails densities ($\alpha \leq 1$). MLE is preferable for $n = 50$. Moreover moments estimator is competitive for small n and big α. According to the stability to the changes of levels of n and α, the L-moments, moments and MLE methods can be rated in this order.

From our study on relative performance of the three different estimation methods in small samples it is clear that no golden rule can be ascribed as to the best one for all parameters in small samples. Note that this in no way contradicts Hosking and Wallis (1987) conclusion that L-method outperforms others, as the performance of this estimator is heavily model dependent in such a situation.

5 Conclusions

This article presents the analytical expression of the first four L-moments of the asymmetric exponential power distributions, making possible considering L-method estimators for the parameters of this distributions family, as an alternative to maximum likelihood and moments estimation. An extensive simulation study has been developed to compare these three estimation methods. It shows that the L-moments method is competitive for small sample sizes and heavy tails distributions.

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			$n=10$			$n=20$			$n=50$		
α	κ		MLE	Mom.	L-Mom.	MLE	Mom.	L-Mom.	MLE	Mom.	L-Mom.
0.50	0.50	$\hat{\alpha}$	1.58	1.45	0.669	0.763	1.46	0.575	0.556	1.53	0.528
			(2.13)	*(2.12)*	*(0.611)*	*(0.776)*	*(2.25)*	*(0.3)*	*(0.16)*	*(2.38)*	*(0.101)*
		$\hat{\sigma}$	0.917	1.08	0.884	0.891	1.06	0.895	0.914	1.03	0.903
			(0.434)	*(0.236)*	*(0.354)*	*(0.427)*	*(0.237)*	*(0.345)*	*(0.368)*	*(0.229)*	*(0.332)*
		$\hat{\kappa}$	0.564	0.637	0.578	0.542	0.632	0.549	0.528	0.624	0.529
			(0.129)	*(0.288)*	*(0.175)*	*(0.0822)*	*(0.292)*	*(0.117)*	*(0.0504)*	*(0.288)*	*(0.0624)*
	1.00	$\hat{\alpha}$	1.63	1.03	0.547	0.759	1.07	0.52	0.554	1.08	0.502
			(2.31)	*(1.57)*	*(0.309)*	*(0.812)*	*(1.71)*	*(0.215)*	*(0.161)*	*(1.78)*	*(0.143)*
		$\hat{\sigma}$	0.912	1.07	0.825	0.906	1.05	0.808	0.911	1.03	0.776
			(0.436)	*(0.229)*	*(0.384)*	*(0.42)*	*(0.223)*	*(0.389)*	*(0.376)*	*(0.201)*	*(0.4)*
		$\hat{\kappa}$	1.04	0.964	0.982	1.02	0.969	1	1.01	0.969	1
			(0.331)	*(0.255)*	*(0.216)*	*(0.209)*	*(0.247)*	*(0.172)*	*(0.121)*	*(0.231)*	*(0.124)*
1.00	0.50	$\hat{\alpha}$	2.88	2.37	1.74	1.82	1.75	1.35	1.17	1.34	1.11
			(2.92)	*(2.1)*	*(1.74)*	*(1.73)*	*(1.32)*	*(1.04)*	*(0.502)*	*(0.635)*	*(0.454)*
		$\hat{\sigma}$	0.924	1.14	0.842	0.939	1.12	0.876	0.969	1.08	0.913
			(0.369)	*(0.215)*	*(0.359)*	*(0.335)*	*(0.215)*	*(0.331)*	*(0.256)*	*(0.221)*	*(0.287)*
		$\hat{\kappa}$	0.549	0.581	0.559	0.537	0.561	0.542	0.522	0.539	0.525
			(0.104)	*(0.151)*	*(0.107)*	*(0.0709)*	*(0.113)*	*(0.0729)*	*(0.0413)*	*(0.0712)*	*(0.0435)*
	1.00	$\hat{\alpha}$	3.26	1.65	2.06	1.89	1.32	1.86	1.18	1.05	1.64
			(3.3)	*(1.66)*	*(2.19)*	*(1.84)*	*(1.24)*	*(1.93)*	*(0.503)*	*(0.737)*	*(1.53)*
		$\hat{\sigma}$	0.958	0.838	0.872	0.972	0.77	0.867	0.98	0.704	0.866
			(0.359)	*(0.36)*	*(0.359)*	*(0.312)*	*(0.389)*	*(0.36)*	*(0.252)*	*(0.416)*	*(0.355)*
		$\hat{\kappa}$	1.04	0.987	0.986	1.01	0.997	0.995	1.01	1.01	1
			(0.308)	*(0.183)*	*(0.185)*	*(0.184)*	*(0.153)*	*(0.144)*	*(0.104)*	*(0.118)*	*(0.0976)*
2.00	0.50	$\hat{\alpha}$	4.22	3.73	3.06	3.65	3.18	2.74	2.62	2.55	2.36
			(3.07)	*(2.52)*	*(2.34)*	*(2.56)*	*(2.03)*	*(1.93)*	*(1.37)*	*(1.25)*	*(1.26)*
		$\hat{\sigma}$	0.871	0.995	0.826	0.94	1	0.878	0.987	1.01	0.946
			(0.327)	*(0.237)*	*(0.336)*	*(0.255)*	*(0.207)*	*(0.284)*	*(0.176)*	*(0.169)*	*(0.202)*
		$\hat{\kappa}$	0.535	0.566	0.556	0.527	0.545	0.538	0.519	0.527	0.523
			(0.0777)	*(0.109)*	*(0.0979)*	*(0.0544)*	*(0.0726)*	*(0.0663)*	*(0.0347)*	*(0.0441)*	*(0.0398)*
	1.00	$\hat{\alpha}$	4.84	2.1	4.02	3.71	1.91	3.87	2.63	1.79	3.72
			(3.45)	*(1.79)*	*(3.09)*	*(2.59)*	*(1.65)*	*(2.99)*	*(1.39)*	*(1.56)*	*(2.87)*
		$\hat{\sigma}$	0.932	0.69	0.897	0.985	0.679	0.921	0.998	0.671	0.941
			(0.284)	*(0.409)*	*(0.314)*	*(0.223)*	*(0.415)*	*(0.303)*	*(0.167)*	*(0.42)*	*(0.291)*
		$\hat{\kappa}$	1.03	0.994	0.989	1.01	0.999	0.998	1	1	1
			(0.259)	*(0.162)*	*(0.169)*	*(0.159)*	*(0.133)*	*(0.131)*	*(0.0894)*	*(0.095)*	*(0.0881)*
4.00	0.50	$\hat{\alpha}$	4.93	4.54	3.97	5.11	4.47	4.04	4.77	4.35	4.12
			(2.06)	*(1.79)*	*(2.09)*	*(1.83)*	*(1.66)*	*(1.87)*	*(1.52)*	*(1.44)*	*(1.6)*
		$\hat{\sigma}$	0.818	0.938	0.809	0.903	0.963	0.875	0.969	0.982	0.941
			(0.308)	*(0.235)*	*(0.324)*	*(0.211)*	*(0.19)*	*(0.252)*	*(0.117)*	*(0.125)*	*(0.153)*
		$\hat{\kappa}$	0.527	0.565	0.552	0.519	0.541	0.533	0.513	0.524	0.521
			(0.0595)	*(0.102)*	*(0.0893)*	*(0.0376)*	*(0.0656)*	*(0.0583)*	*(0.0243)*	*(0.0409)*	*(0.0375)*
	1.00	$\hat{\alpha}$	5.52	2.84	4.65	5.16	2.73	4.68	4.8	2.61	4.64
			(2.04)	*(2.44)*	*(2.1)*	*(1.87)*	*(2.49)*	*(2.02)*	*(1.54)*	*(2.51)*	*(1.8)*
		$\hat{\sigma}$	0.868	0.711	0.904	0.939	0.701	0.922	0.983	0.693	0.947
			(0.256)	*(0.389)*	*(0.272)*	*(0.17)*	*(0.391)*	*(0.237)*	*(0.102)*	*(0.393)*	*(0.193)*
		$\hat{\kappa}$	1.03	0.999	0.995	1.01	1	1	1	1	1
			(0.228)	*(0.153)*	*(0.162)*	*(0.125)*	*(0.12)*	*(0.126)*	*(0.0706)*	*(0.0794)*	*(0.0842)*

Table 1: Simulations results. Known θ.

α	κ		$n=10$ MLE	$n=10$ Mom.	$n=10$ L-Mom.	$n=20$ MLE	$n=20$ Mom.	$n=20$ L-Mom.	$n=50$ MLE	$n=50$ Mom.	$n=50$ L-Mom.
0.50	0.50	$\hat{\alpha}$	1.3	1.34	0.555	0.672	1.43	0.533	0.518	1.42	0.504
			(2.08)	*(2.07)*	*(0.426)*	*(0.857)*	*(2.24)*	*(0.34)*	*(0.159)*	*(2.26)*	*(0.173)*
		$\hat{\sigma}$	1.11	1.53	1.14	1.1	1.44	1.13	1.11	1.34	1.06
			(0.695)	*(0.772)*	*(0.711)*	*(0.669)*	*(0.719)*	*(0.694)*	*(0.617)*	*(0.639)*	*(0.637)*
		$\hat{\kappa}$	0.65	0.585	0.647	0.608	0.555	0.593	0.567	0.51	0.547
			(0.3)	*(0.389)*	*(0.3)*	*(0.218)*	*(0.373)*	*(0.229)*	*(0.137)*	*(0.35)*	*(0.126)*
		$\hat{\theta}$		2.1			1.33			0.767	
				(4)			*(2.48)*			*(1.38)*	
	1.00	$\hat{\alpha}$	1.2	0.998	0.559	0.616	1.04	0.532	0.509	1.06	0.512
			(1.94)	*(1.51)*	*(0.312)*	*(0.736)*	*(1.67)*	*(0.219)*	*(0.163)*	*(1.76)*	*(0.123)*
		$\hat{\sigma}$	1.06	1.53	1.22	1.01	1.41	1.18	1.05	1.28	1.13
			(0.676)	*(0.763)*	*(0.75)*	*(0.639)*	*(0.683)*	*(0.738)*	*(0.588)*	*(0.582)*	*(0.719)*
		$\hat{\kappa}$	0.959	0.909	0.969	0.984	0.917	0.987	0.995	0.92	0.997
			(0.245)	*(0.37)*	*(0.246)*	*(0.184)*	*(0.355)*	*(0.19)*	*(0.134)*	*(0.346)*	*(0.137)*
		$\hat{\theta}$		-0.0188			-0.000895			0.00211	
				(2.25)			*(1.32)*			*(0.747)*	
1.00	0.50	$\hat{\alpha}$	2.44	1.74	1.38	1.69	1.42	1.16	1.12	1.15	0.962
			(2.86)	*(1.51)*	*(1.43)*	*(1.86)*	*(1.06)*	*(1.01)*	*(0.548)*	*(0.522)*	*(0.58)*
		$\hat{\sigma}$	1.03	1.53	1.05	1.09	1.4	0.995	1.12	1.24	0.891
			(0.618)	*(0.755)*	*(0.633)*	*(0.596)*	*(0.685)*	*(0.585)*	*(0.484)*	*(0.557)*	*(0.477)*
		$\hat{\kappa}$	0.682	0.727	0.71	0.653	0.676	0.665	0.613	0.628	0.615
			(0.341)	*(0.325)*	*(0.324)*	*(0.275)*	*(0.257)*	*(0.269)*	*(0.19)*	*(0.184)*	*(0.192)*
		$\hat{\theta}$		0.656			0.52			0.373	
				(1.08)			*(0.844)*			*(0.588)*	
	1.00	$\hat{\alpha}$	2.5	1.77	1.96	1.63	1.52	2.03	1.09	1.25	2
			(2.92)	*(1.71)*	*(2.02)*	*(1.78)*	*(1.46)*	*(1.99)*	*(0.509)*	*(1.01)*	*(1.82)*
		$\hat{\sigma}$	0.971	1.12	1.24	1	1.04	1.34	0.981	0.97	1.41
			(0.584)	*(0.6)*	*(0.694)*	*(0.542)*	*(0.597)*	*(0.737)*	*(0.398)*	*(0.596)*	*(0.764)*
		$\hat{\kappa}$	0.942	0.968	0.961	0.976	0.983	0.98	0.996	0.996	0.996
			(0.269)	*(0.227)*	*(0.235)*	*(0.204)*	*(0.191)*	*(0.195)*	*(0.151)*	*(0.15)*	*(0.148)*
		$\hat{\theta}$		0.000593			-0.0086			0.00371	
				(0.62)			*(0.472)*			*(0.326)*	
2.00	0.50	$\hat{\alpha}$	3.66	2.83	2.55	3.37	2.54	2.46	2.61	2.03	2.13
			(3.04)	*(2.03)*	*(2.12)*	*(2.58)*	*(1.85)*	*(2.01)*	*(1.54)*	*(1.33)*	*(1.67)*
		$\hat{\sigma}$	0.912	1.12	1.03	1.05	1.06	1.01	1.13	0.976	0.964
			(0.453)	*(0.463)*	*(0.551)*	*(0.42)*	*(0.417)*	*(0.508)*	*(0.333)*	*(0.359)*	*(0.433)*
		$\hat{\kappa}$	0.757	0.8	0.812	0.734	0.758	0.779	0.695	0.705	0.732
			(0.415)	*(0.395)*	*(0.411)*	*(0.361)*	*(0.346)*	*(0.379)*	*(0.291)*	*(0.282)*	*(0.327)*
		$\hat{\theta}$		0.565			0.503			0.408	
				(0.833)			*(0.73)*			*(0.587)*	
	1.00	$\hat{\alpha}$	3.76	2.58	3.64	3.38	2.68	3.98	2.5	2.83	4.26
			(3.09)	*(1.93)*	*(2.77)*	*(2.58)*	*(2.03)*	*(2.95)*	*(1.4)*	*(2.17)*	*(3.07)*
		$\hat{\sigma}$	0.792	0.836	1.02	0.895	0.855	1.06	0.951	0.88	1.1
			(0.391)	*(0.388)*	*(0.472)*	*(0.324)*	*(0.362)*	*(0.413)*	*(0.221)*	*(0.343)*	*(0.345)*
		$\hat{\kappa}$	0.93	0.957	0.955	0.96	0.971	0.964	0.986	0.985	0.982
			(0.3)	*(0.247)*	*(0.249)*	*(0.241)*	*(0.223)*	*(0.231)*	*(0.19)*	*(0.188)*	*(0.199)*
		$\hat{\theta}$		-0.000172			-0.000527			0.00682	
				(0.464)			*(0.4)*			*(0.322)*	
4.00	0.50	$\hat{\alpha}$	4.35	3.21	3.39	4.72	3.42	3.55	4.64	3.58	3.65
			(2.4)	*(2.07)*	*(2.24)*	*(2.02)*	*(1.98)*	*(2.13)*	*(1.63)*	*(1.9)*	*(1.97)*
		$\hat{\sigma}$	0.829	1	1.05	0.99	1.03	1.07	1.1	1.04	1.08
			(0.364)	*(0.369)*	*(0.495)*	*(0.29)*	*(0.336)*	*(0.442)*	*(0.237)*	*(0.312)*	*(0.389)*
		$\hat{\kappa}$	0.82	0.871	0.876	0.801	0.835	0.844	0.787	0.809	0.821
			(0.473)	*(0.461)*	*(0.469)*	*(0.438)*	*(0.432)*	*(0.445)*	*(0.403)*	*(0.408)*	*(0.424)*
		$\hat{\theta}$		0.618			0.558			0.513	
				(0.857)			*(0.78)*			*(0.711)*	
	1.00	$\hat{\alpha}$	4.47	3.19	3.98	4.77	3.44	4.32	4.65	3.91	4.72
			(2.39)	*(2.09)*	*(2.11)*	*(2.02)*	*(2.02)*	*(2.01)*	*(1.61)*	*(1.9)*	*(1.87)*
		$\hat{\sigma}$	0.706	0.803	0.935	0.814	0.829	0.969	0.906	0.885	1.01
			(0.364)	*(0.326)*	*(0.4)*	*(0.272)*	*(0.286)*	*(0.35)*	*(0.162)*	*(0.226)*	*(0.277)*
		$\hat{\kappa}$	0.905	0.944	0.941	0.94	0.955	0.951	0.959	0.96	0.954
			(0.335)	*(0.269)*	*(0.271)*	*(0.284)*	*(0.256)*	*(0.262)*	*(0.242)*	*(0.24)*	*(0.248)*
		$\hat{\theta}$		0.0014			0.0119			0.0038	
				(0.461)			*(0.426)*			*(0.38)*	

Table 2: Simulations results. Unknown θ.

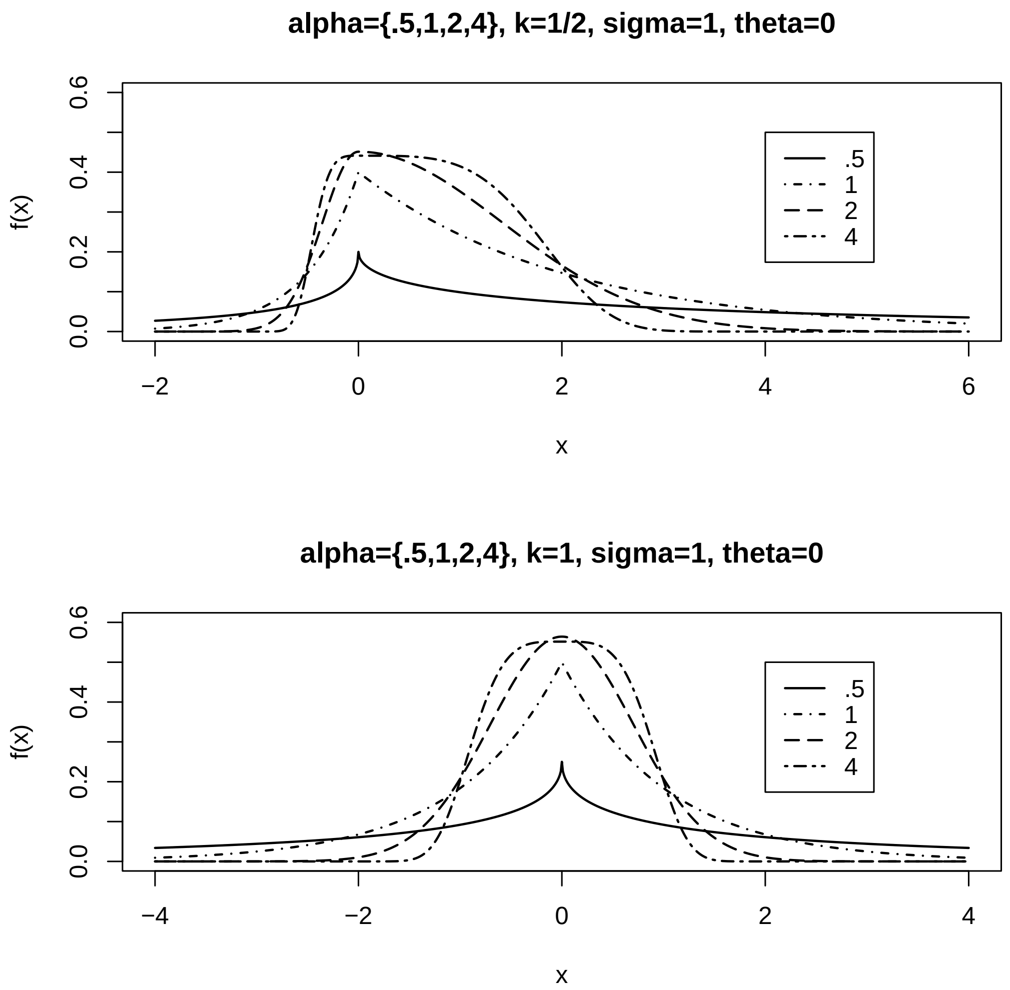


Figure 1: Probability density function of asymmetric exponential power distributions for some parameter combinations.

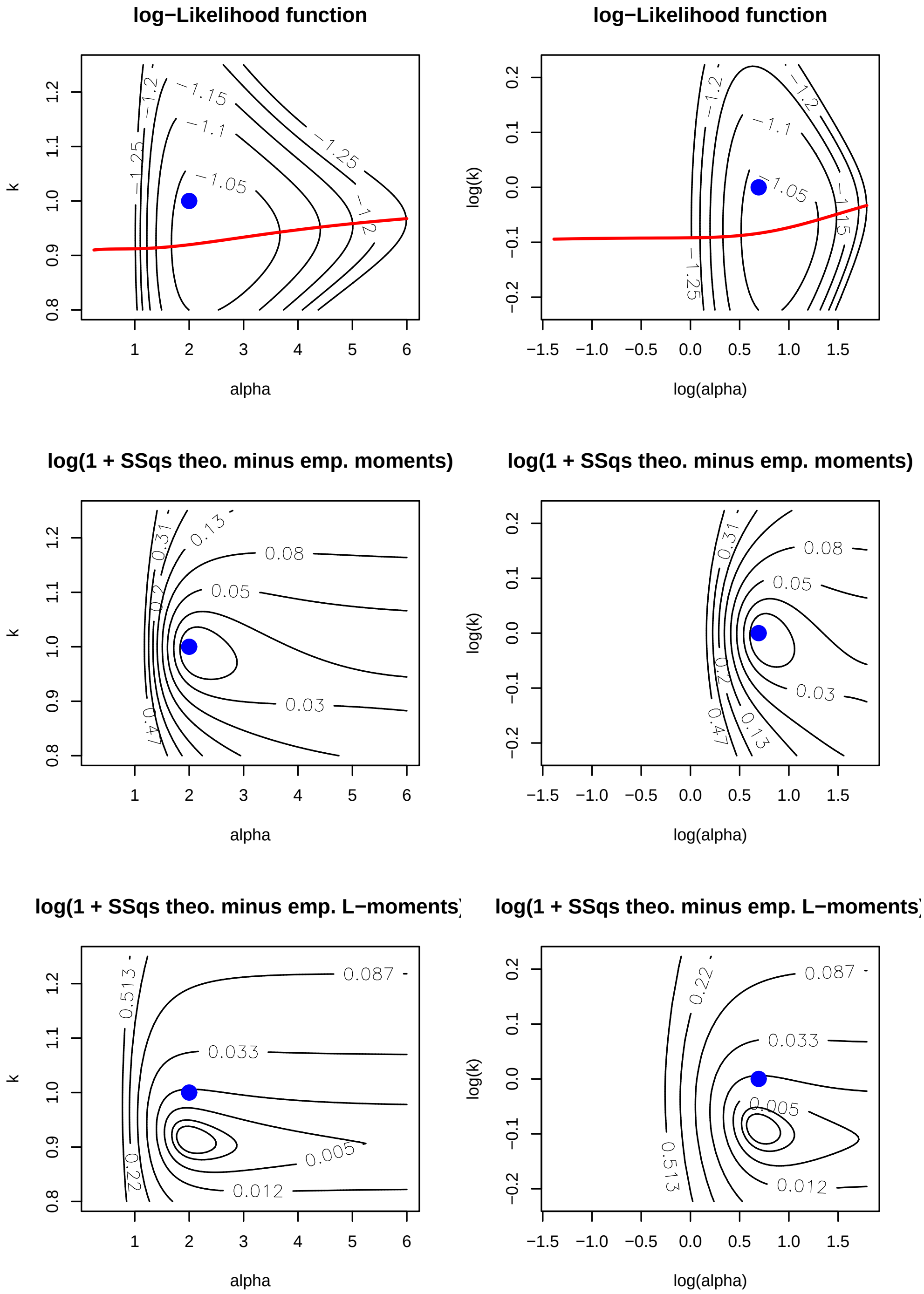


Figure 2: Contour level plots for the objective functions optimized in the estimation process. The big dot is placed in the parameter values used to generate the data. The theoretical expression of the MLE of κ as a function of α (see equation 2) is represented in the first row of graphics.

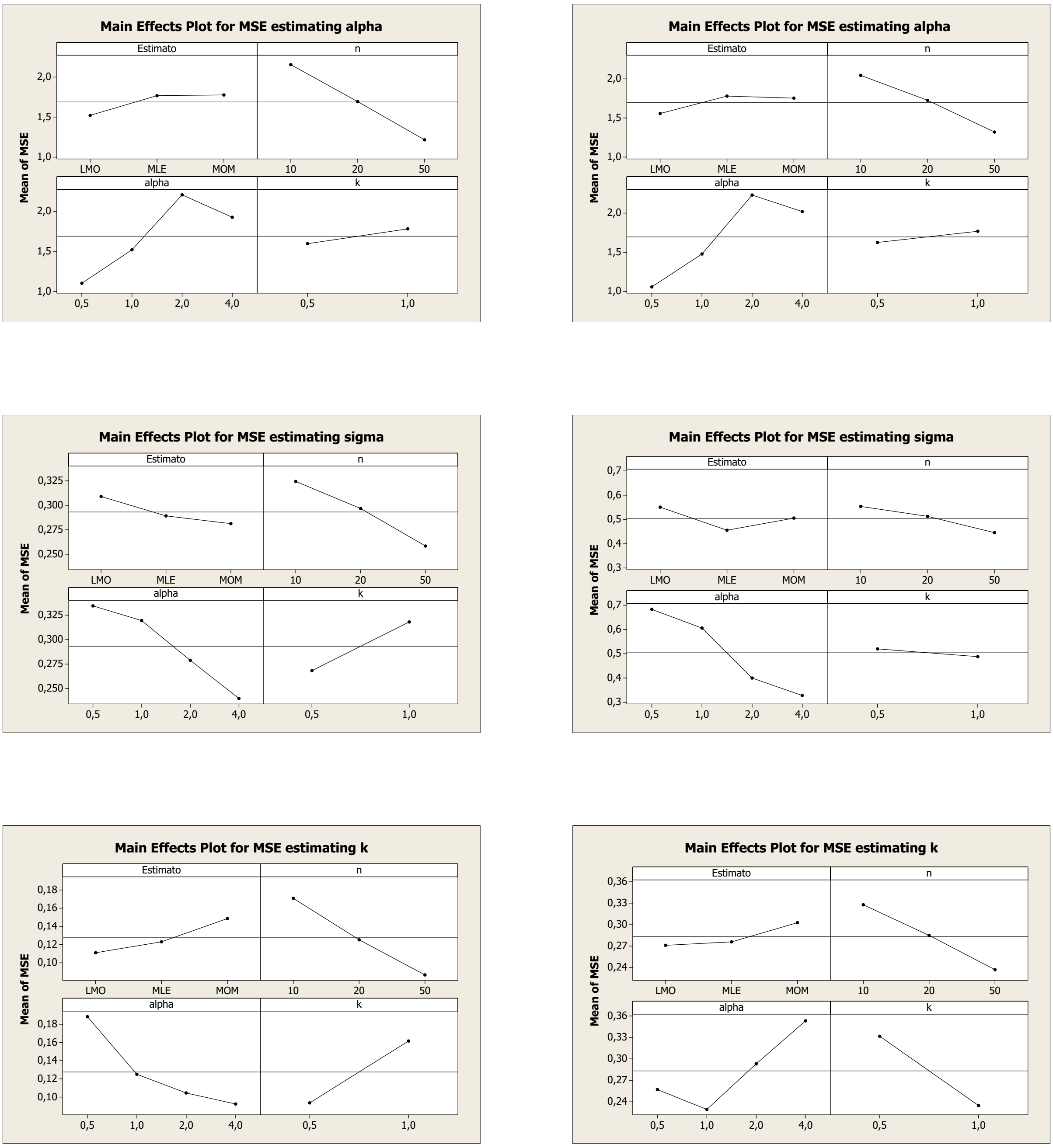
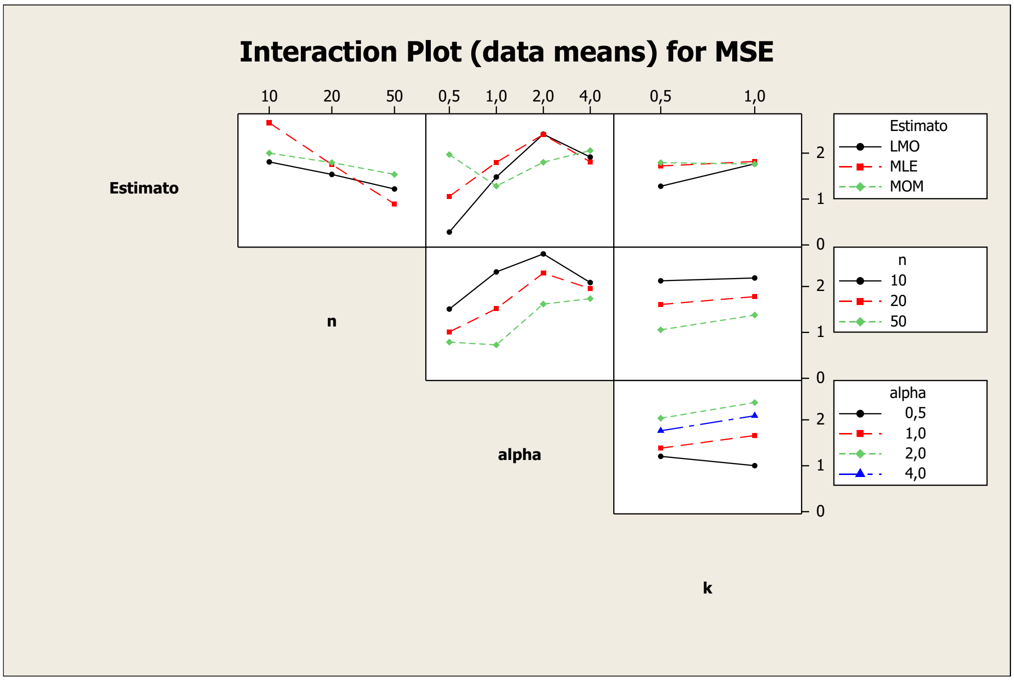


Figure 3: Main effects plots for the square root of MSE values from Tables 1 (known θ, left column) and 2 (estimated θ, right column). First row of graphics represents the square root of MSE in the estimation of α, second row corresponds to the estimation of σ, and the third one to the estimation of κ.



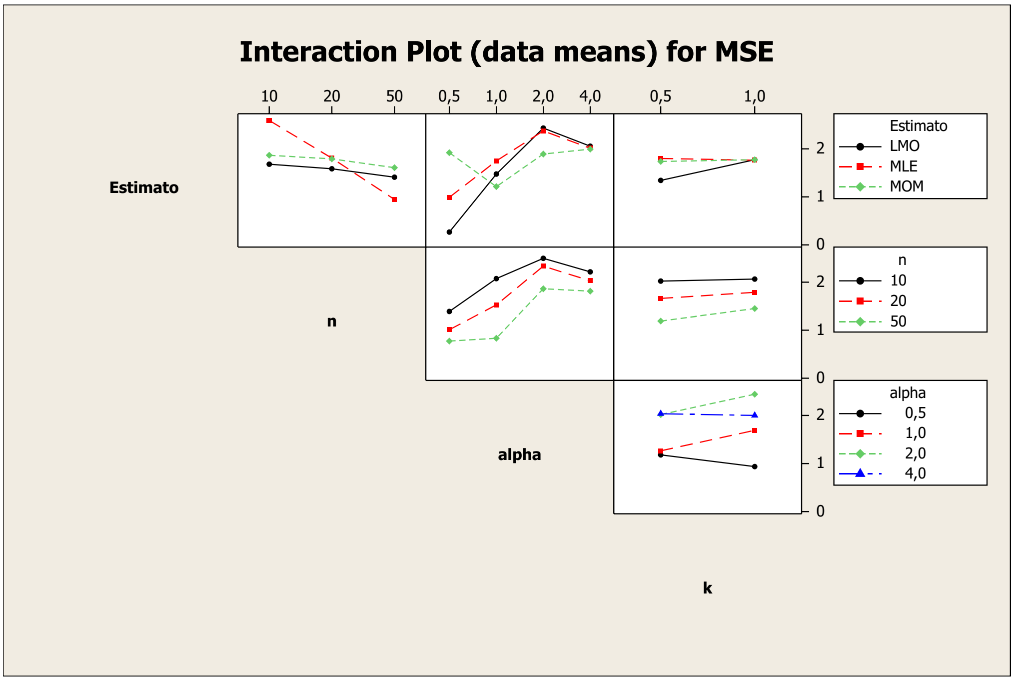
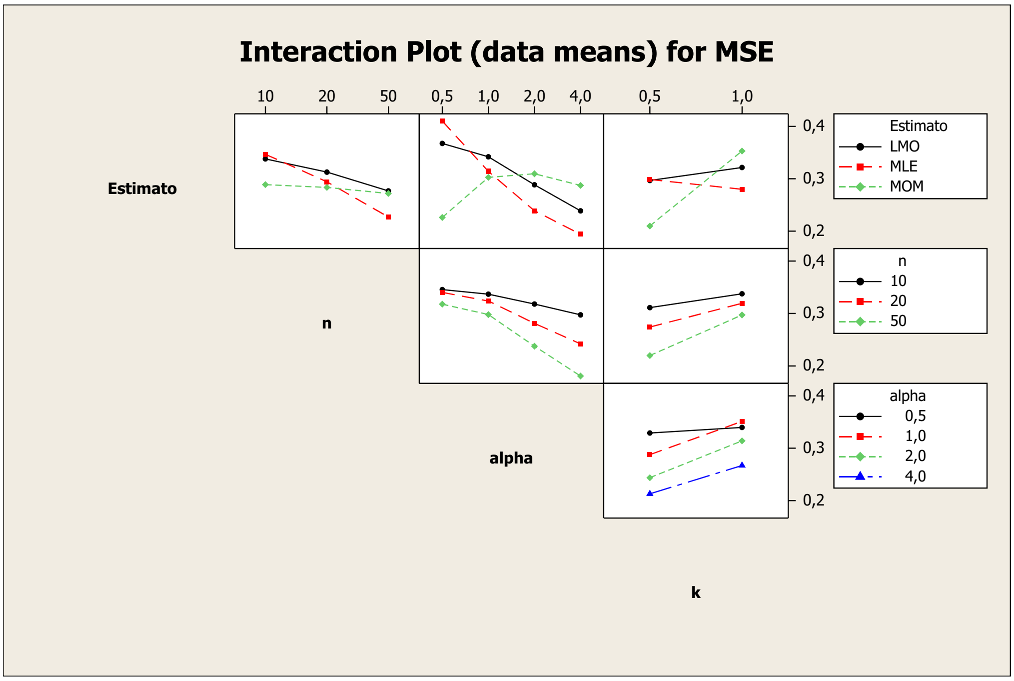


Figure 4: Estimation of α. Interaction plot for the MSE values. Top panel: known θ. Bottom panel: estimated θ.



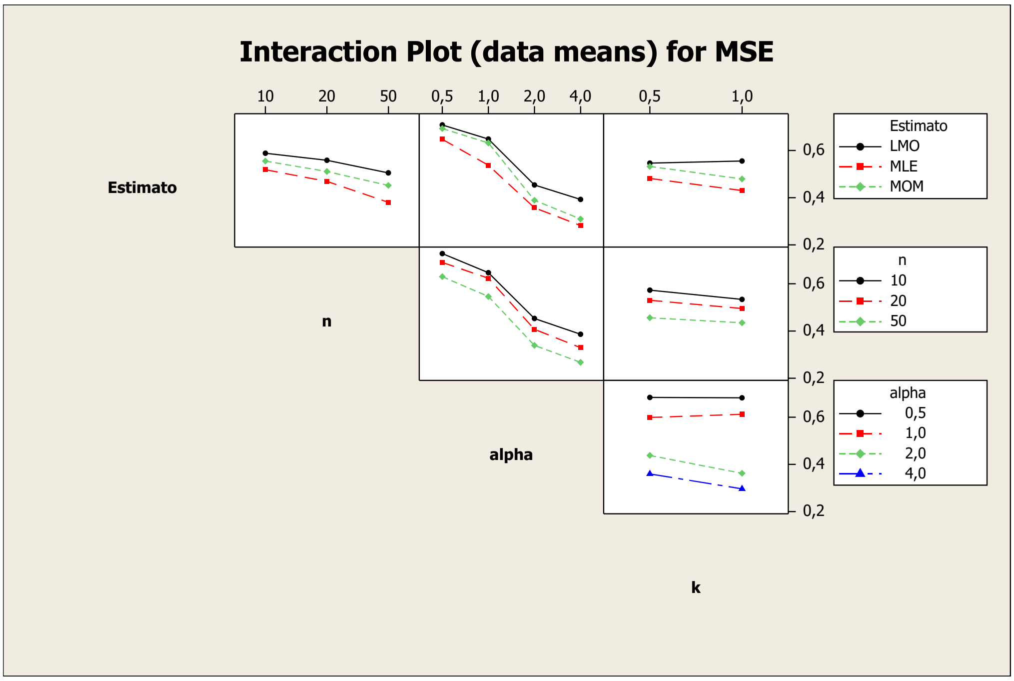
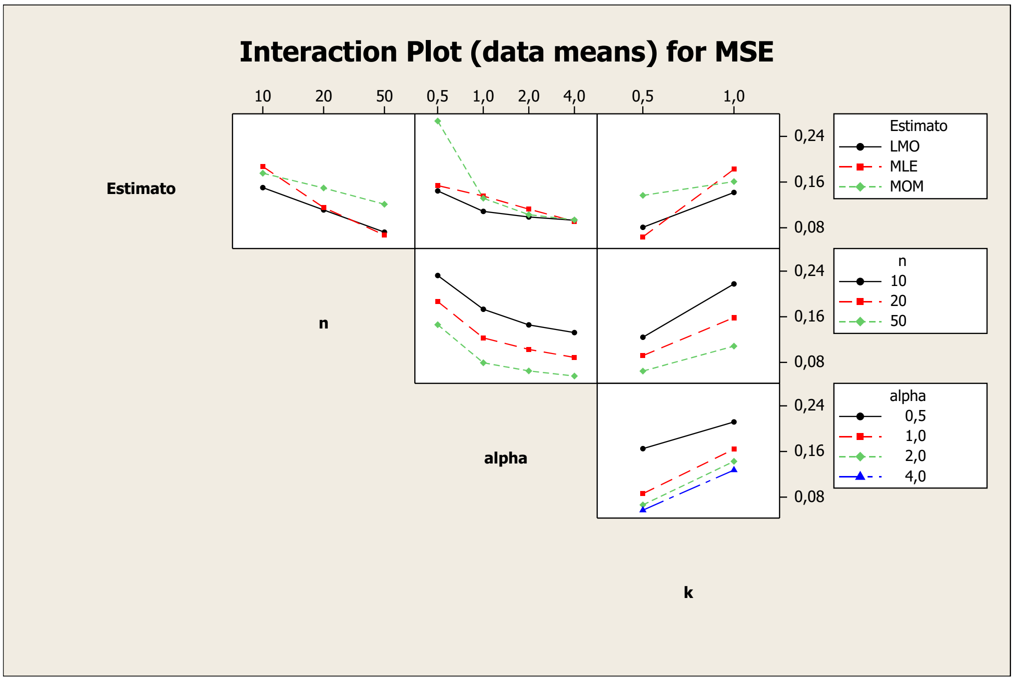


Figure 5: Estimation of σ. Interaction plot for the MSE values. Top panel: known θ. Bottom panel: estimated θ.



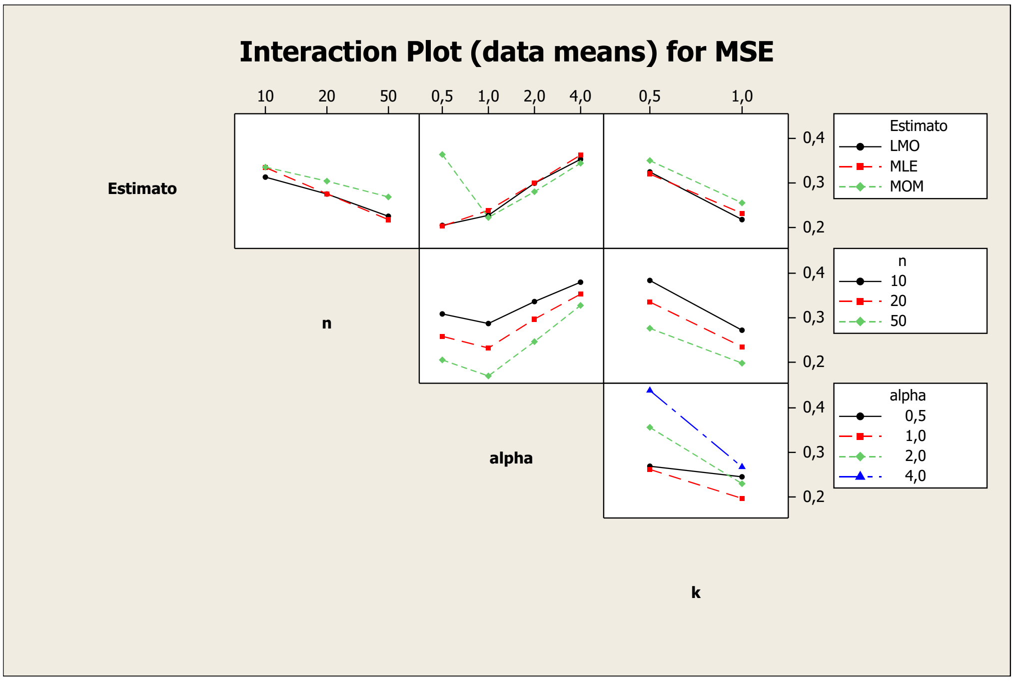


Figure 6: Estimation of κ. Interaction plot for the MSE values. Top panel: known θ. Bottom panel: estimated θ.