The radar multiplier method: a two-phase approach for large scale nonlinear combinatorial optimization problems *

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In this paper we consider the following class of nonlinear combinatorial optimization problems:

$$\left.\begin{array}{ll} \min & f(x) \\ s.t. & x \in \mathcal{D} \cap \widetilde{\mathcal{D}} \end{array}\right\}, \tag{1}$$

where f is a general nonlinear function $f : \Re^n \rightarrow \Re$, $\mathcal{D}$ is compact, connected set, and $\widetilde{\mathcal{D}}$ is a compact set imposing integrality conditions over the variables x (that is, $\widetilde{\mathcal{D}}$ is a compact non-connected set). It is well known that problems like (1) are specially appropiate to be solved through the so called Variable Duplication (VD) technique, where the original problem (1) is transformed into the following equivalent problem expression:

$$\left.\begin{array}{ll} \min & \hat{f}(x) + \widetilde{f}(\widetilde{x}) \\ s.t. & x \in \mathcal{D}, \quad \widetilde{x} \in \widetilde{\mathcal{D}} \\ & x - \widetilde{x} = 0 \end{array}\right\}, \tag{2}$$

where $f(x) = \hat{f}(x) + \widetilde{f}(x)$ ($\hat{f}(x)$ and $\widetilde{f}$ could be just $\hat{f}(x) = \widetilde{f}(x) = 1/2 f(x)$ or any other more elaborated decomposition of $f(x)$). The Lagrangian Relaxation (LR) method is a powerful technique to solve this kind of problems. When LR is applied to eliminate the coupling constraints $x - \widetilde{x} = 0$, two approaches can be followed: the Classical Lagrangian Relaxation (CLR) method and the Augmented Lagrangian Relaxation (ALR) method. Both methods have been described, studied, developed and tested extensively.

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The CLR method has two main drawbacks. First, this method induces a dual function which is not necessarily differentiable. The subgradient method was the pioneering method used to treat this dual function. Alternatives to the subgradient method are bundle method [HUL96], or Radar Subgradient (RS) method [BH99], which notably improve the subgradient method. The second drawback is that the CLR method usually gives primal infeasible solutions, so solutions must be processed through heuristics after optimization in order to gain feasibility.

The ALR method also has two main drawbacks. Firstly, we lose the separability of the problem because of the added quadratic term. The Block Coordinated Descent (BCD) [BH00] or the Auxiliary Problem Principle (APP) [Coh80] can be used to cope with the non-separable augmented Lagrangian. Secondly, as it will be illustrated in this paper, the ALR method gives local optimizers. Furthermore, within the ALR method, we need some kind of lower bound to the optimal cost in order to control the quality of the computed local optimizers.

Instead of choosing between the CLR and the ALR methods, and having to deal with the inherited drawbacks (primal infeasibility and lack of a cost lower bound) we propose to use both methods complementarily. The CLR method will provide the ALR method with a cost lower bound and the ALR method will process the primal infeasible solution obtained by the CLR method towards feasibility. Thus, we propose a two-phase procedure to solve the problem. In the first phase the CLR method is applied and produces a primal-dual solution pair $(\widehat{x}, \widehat{\lambda})$. Then, unless $\widehat{x}$ is feasible, the second phase is activated and, the ALR method starting from $(\widehat{x}, \widehat{\lambda})$ searches for a new optimal pair (x^*, λ^*) with a primal feasible x^*. The sketch of the proposed algorithm, called the *radar multiplier* (RM) method is:

Phase 1 [Compute a cost lower bound $\underline{f}^*$.]

Using the radar subgradient method solve

$$\max_{\lambda \in R^n} \left\{ \min_{\substack{x \in \mathcal{D} \\ \widetilde{x} \in \widetilde{\mathcal{D}}}} f(x) + \widetilde{f}(\widetilde{x}) + \lambda'(x - \widetilde{x}) \right\}. \tag{3}$$

Phase 2 [Compute the local optimizer $(x^*, \widetilde{x}^*)$.]

Using the augmented Lagrangian relaxation method (block coordinated descent version) solve

$$\max_{\lambda \in R^n} \left\{ \min_{\substack{x \in \mathcal{D} \\ \widetilde{x} \in \widetilde{\mathcal{D}}}} f(x) + \widetilde{f}(\widetilde{x}) + \lambda'(x - \widetilde{x}) + \tfrac{c}{2}\|x - \widetilde{x}\|^2 \right\}. \tag{4}$$

An alternative to the RM method could be the Subgradient Multiplier (SM) method if we use the subgradient method in phase 1, or the bundle multiplier method, and so on. The advantages of this approach over an heuristic method are: (1) the ALR method reaches a (local) optimizer while a heuristic method may obtain a (local) suboptimal solution, (2) the ALR method does not depend on the problem solved whereas a heuristic method must be designed according to the model used, (3) furthermore, the ALR method can be used to reach feasibility in a great variety of combinatorial problems whereas an heuristic method, if possible to use, should be adapted for each new problem.

This paper is structured in three sections:

- In section 1 we illustrate that the ALR method may give local optimizers of very poor quality. We present a test of 10 examples where the ALR method reaches a global optimizer for 2 cases whereas in the other 8 cases the ALR method stopped at a local optimizer.
- in section 2 we show how this situation can be notably improved by tuning the penalty parameter c_n properly. Empirically, the best results are obtained by starting with a small c_0 and gradually increasing it to reach the differentiability of the dual function.
- Finally, in the the third section the Radar Subgradient (RS) method and the ALR method (also denoted as multiplier method) are combined to yield the Radar Multiplier (RM) method. The method is described and tested over 18 instances of the Unit Commitment (UC) problem (a particular case of problem (1) arising in electrical engineering) ranging from 16 continuous variables and 4 binary variables to 9408 continuous variables and 1848 binary variables. Within the run tests, the RM method has shown to be an effective method to obtain very high quality solutions (with a very low duality gap) in a reasonable CPU time.

References

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