# Conic quadratic optimization - part 2

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# Introduction

# Topics

- Duality in conic optimization.
- Optimality conditions for conic quadratic optimization problems.

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# Duality

# The (dirty) details of duality



Recall given a convex cone $\mathcal{K}$ then the dual cone $\mathcal{K}^*$ is given by

$$\mathcal{K}^* := \{s : \; s^T x \geq 0, \; \forall x \in \mathcal{K}\}.$$

Moreover, the primal conic optimization

$$\begin{array}{lllll} \min & \sum_{k=1}^{r} (c^k)^T x^k & & & \\ \text{st} & \sum_{k=1}^{r} A^k x^k & = & b, & \\ & x^k \in \mathcal{K}^k, & & & k = 1, \ldots, r, \end{array} \tag{1}$$

has the corresponding dual problem

$$\begin{array}{lllll} \max & b^T y & & & \\ \text{st} & (A^k)^T y + s^k & = & c^k, & k = 1, \ldots, r, \\ & s^k \in (\mathcal{K}^k)^*, & & & k = 1, \ldots, r. \end{array} \tag{2}$$

# Simplifying notation



Let us simplify the notation i.e.

$$\begin{array}{rcl} c & := & \begin{bmatrix} c^1 \\ c^2 \\ \vdots \\ c^r \end{bmatrix}, \\ A & := & \begin{bmatrix} A^1 & A^2 & \ldots & A^r \end{bmatrix}, \\ \mathcal{K} & := & \mathcal{K}^1 \times \mathcal{K}^2 \times \cdots \times \mathcal{K}^r, \\ \mathcal{K}^* & := & (\mathcal{K}^1)^* \times (\mathcal{K}^2)^* \times \cdots \times (\mathcal{K}^r)^*. \end{array}$$

We assume that

$$A \in \mathcal{R}^{m \times n}.$$

Moreover let

$$x := \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^r \end{bmatrix} \text{ and } s := \begin{bmatrix} s^1 \\ s^2 \\ \vdots \\ s^r \end{bmatrix}.$$

Observe that

$$\mathcal{K}^* = \{s \in \mathcal{R}^n : \; x^T s \geq 0, \; \forall x \in \mathcal{K}\}.$$

The primal problem

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{st} & Ax & = & b, \\ & x \in \mathcal{K}. & & \end{array} \tag{3}$$

and the dual problem

$$\begin{array}{lrcl} \max & b^T y & & \\ \text{st} & A^T y + s & = & c, \\ & s \in \mathcal{K}^*. & & \end{array} \tag{4}$$

Definitions:

- The problem is *primal feasible* if a solution $x$ exists satisfying the constraints of (3).
- The problem is *dual feasible* if a solution $(y, s)$ exists satisfying the constraints of (4).

## Weak duality

Given a primal-dual feasible solution $(x, y, s)$ then

$$
\begin{array}{rcl}
\text{duality gap} & := & c^T x - b^T y \\
 & = & (A^T y + s)^T x - b^T y \\
 & = & x^T s + (Ax)^T y - b^T y \\
 & = & x^T s + b^T y - b^T y \\
 & = & x^T s \\
 & \geq & 0.
\end{array}
$$

Recall $x \in \mathcal{K}$ and $s \in \mathcal{K}^*$ implies $x^T s \geq 0$.
Hence for all primal-dual feasible solutions $(x, y, s)$ weak duality holds i.e.

$$c^T x \quad \geq \quad b^T y,$$

or in words

$$\text{primal objective value} \quad \geq \quad \text{dual objective value.}$$

**Lemma 1** *If $(x, y, s)$ is a primal-dual feasible solution and*

$$c^T x = b^T y,$$

*then $x$ is an optimal solution to (3) and $(y, s)$ is an optimal solution to (4).*

Notes:

- The value of weak duality cannot be overstated.
  - Makes it possible to state a certificate that allows verification of optimality in polynomial time.
  - Makes it possible to evaluate the quality of feasible solution given a dual feasible solution known.

# Questions and answers



## Does (3) always have an optimal solution?

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No the problem could be infeasible e.g.

$$
\begin{array}{lll}
\min & x & \\
\text{st} & x & = -1, \\
& x \geq 0. &
\end{array}
$$

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No. For instance the problem

$$\begin{array}{lll} \min & -x_2 & \\ \text{st} & x_1 - x_3 & = \ 0, \\ & \sqrt{x_2^2 + x_3^2} \leq x_1, & \end{array}$$

has the set feasible solutions:

$$\{(x_1, x_2, x_3) : \ x_1 \geq 0, \ x_2 = 0, \ x_3 \geq 0\}.$$

Hence, $x = (0, 0, 0)$ is an optimal solution.

The corresponding dual problem is

$$
\begin{array}{lccc}
\max & 0 & & \\
\text{st} & y + s_1 & = & 0, \\
 & s_2 & = & -1, \\
 & -y + s_3 & = & 0, \\
 & \sqrt{s_2^2 + s_3^2} \leq s_3. & &
\end{array}
$$

Hence,

$$
\sqrt{s_1^2 + 1} \leq s_1
$$

which implies the dual problem is infeasible.

**Assume both (3) and (4) has an optimal solution is the duality gap then zero?**

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No, consider

$$\begin{array}{lrcl} \min & x_2 & & \\ \text{st} & \sqrt{x_1^2 + (x_2 - 1)^2} & \leq & x_1, \\ & \sqrt{(-x_1 + x_2)^2} & \leq & x_1. \end{array}$$

From the first constraint it follows

$$x_2 = 1$$

Using this fact and the second constraint then

$$1 \leq 2x_1.$$

The set of primal feasible solutions is

$$\left\{(x_1, x_2): \; x_1 \geq \frac{1}{2}, \; x_2 = 1\right\}$$

and the optimal objective value is $1$.
The corresponding dual problem is

$$\begin{array}{lccc}
\max & z_2 & & \\
\text{st} & z_1 + w_1 - z_3 + w_2 & = & 0, \\
 & z_2 + z_3 & = & 1, \\
 & \sqrt{z_1^2 + z_2^2} & \leq & w_1, \\
 & \sqrt{z_3} & \leq & w_2.
\end{array}$$

The two last constraints implies

$$w_1 \geq |z_1| \text{ and } w_2 \geq |z_3|$$

we have

$$w_1 + z_1 \geq 0 \text{ and } w_2 - z_3 \geq 0.$$

Using the first constraint this implies

$$w_1 = -z_1 \text{ and } w_2 = z_3.$$

Now using the second constraint we have that

$$z_2 = 1 - z_3 = 1 - w_2.$$

Therefore, the dual problem is equivalent to

$$\begin{array}{lccc} \max & 1 - w_2 & & \\ \text{st} & \sqrt{w_1^2 + (1 - w_2)^2} & \leq & w_1, \\ & \sqrt{w_2^2} & \leq & w_2 \end{array}$$

which has the feasible set $\{(w_1, w_2) : \; w_1 \geq 0, \; w_2 = 1\}$ and the optimal objective value is zero. Hence,

$$\begin{array}{rcl} \text{duality gap} & = & 1 - 0 \\ & = & 1. \end{array}$$

It can be verified that if

$$(x_2 - 1)^2$$

with

$$(x_2 - \alpha)^2$$

where $\alpha > 0$ then the dual gap will be $\alpha$.

## Is the optimal objective value always attained?

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No. Consider

$$\begin{array}{ll} \min & \frac{1}{x} \\ \text{st} & x \geq 0 \end{array}$$

which is equivalent to

$$\begin{array}{llll} \min & z & & \\ \text{st} & t & = & \sqrt{2}, \\ & t^2 & \leq & 2xz, \quad x, z \geq 0. \end{array}$$

Clearly, the optimal objective value is $0$ but this implies

$$x = \infty$$

which can never be attained.

# Comparison of linear and conic duality



- Almost identical.
- Linear optimization:
  - A nonzero duality gap cannot occur.
  - Problems with attainment does not exist.

# The main theorem

First define the primal problem

$$\begin{array}{rcllll} \nu_p & = & \inf & c^T x & & \\ & & \text{st} & Ax & = & b, \\ & & & x \in \mathcal{K} & & \end{array} \tag{5}$$

and the dual problem

$$\begin{array}{rcllll} \nu_d & = & \sup & b^T y & & \\ & & \text{st} & A^T y + s & = & c, \\ & & & s \in \mathcal{K}^*. & & \end{array} \tag{6}$$

By convention we use

- If (5) is infeasible, then $\nu_p = \infty$.
- If (6) is infeasible, then $\nu_d = -\infty$.

## Strongly feasible



(5) is said to be strongly **feasible** if there $\exists \varepsilon > 0$ such that

$$\{x \in \mathcal{R}^n : \ Ax = \hat{b}, \ x \in \mathcal{K}\} \neq \emptyset$$

for all $\hat{b}$ satisfying

$$\left\| \hat{b} - b \right\| \leq \varepsilon.$$

This is the same as saying that a small perturbation in $b$ does NOT make the problem infeasible.

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# Central statements

**Lemma 2** *(5) is strongly infeasible if and only if*

$$b^T y = 1,\ A^T y + s = 0,\ s \in \mathcal{K}^*$$

*is strongly feasible.*

**Lemma 3** *(6) is strongly infeasible if and only if*

$$c^T x = -1,\ Ax = 0,\ x \in \mathcal{K}$$

*is strongly feasible.*

**Theorem 1** *(Strong duality) If either (5) or (6) is strong feasible, then $\nu_d = \nu_p$.*

See [3, p. 73] and [1].

Observe:

- If $A$ is of full row rank and

$$\operatorname{int}(\{x \in \mathcal{R}^n : \; Ax = b, \; x \in \mathcal{K}\}) \neq \emptyset$$

  then (5) is strongly feasible.
- When does it go wrong?
  - If a small perturbation in the problem data makes the problem status flip from feasible to infeasible or from infeasible to feasible.
- Such problems must be intrinsically hard to solve.
  - Consider that computations are done in finite precision.
  - Data are inaccurate usually.

# The quadratic cone again

# Topological properties



The interior of the quadratic cone is given by

$$\text{int}(\mathcal{K}_q) := \{x \in \mathcal{R}^n : \; x_1 > \|x_{2:n}\|\}.$$

(obviously).

# Complementarity



In linear optimization complementarity means something like

$$x_i s_i = 0.$$

What does the complementarity conditions look like for conic quadratic optimization?
First define the arrow head matrix

$$V := \text{mat}\,(v) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_1 & & \\ \vdots & & \ddots & \\ v_n & & & v_1 \end{bmatrix}.$$

## Observe

$$
\begin{aligned}
\operatorname{mat}(x)\, s &= \begin{bmatrix} x_1 & x_{2:n} & \cdots & x_n \\ x_2 & x_1 & & \\ \vdots & & \ddots & \\ x_n & & & x_1 \end{bmatrix} s \\
&= \begin{bmatrix} x^T s \\ x_1 s_2 + s_1 x_2 \\ \vdots \\ x_1 s_n + s_1 x_n \end{bmatrix}
\end{aligned}
$$

# Complementarity for the quadratic cone



**Lemma 4** *Assume* $\mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^r$ *and each* $\mathcal{K}^k$ *is a quadratic cone. If* $x, s \in \mathcal{K}$, *then* $x$ *and* $s$ *are complementary, i.e.* $x^T s = 0$, *if and only if*

$$X^k S^k e^k = S^k X^k e^k = 0, \quad k = 1, \ldots, r,$$

*where* $X^k := mat\left(x^k\right)$, $S^k := mat\left(s^k\right)$ *and* $e^k = (0, 0, \ldots, k, \ldots, 0)^T \in \mathbf{R}^{n^k}$.

Proof:
Clearly

$$X^k S^k e^k = 0 \Rightarrow (x^k)^T s^k = 0$$

because

$$\begin{aligned} 0 &= \sum_{i=1}^{n} (e^k)^T X^k S^k e^k \\ &= \sum_{i=1}^{k} (x^k)^T s^k \\ &= x^T s. \end{aligned}$$

Next we prove if

$$(x^k)^T s^k = 0 \Rightarrow X^k S^k e^k = 0$$

This is clearly true if $x_1^k = 0$ or $s_1^k = 0$. Therefore, we can assume that $x_1^k > 0$ and $s_1^k > 0$.

Now

$$\begin{aligned} 0 &= x^T s \\ &= \sum_{k=1}^{r} (x^k)^T (s^k) \\ &= \sum_{k=1}^{r} \left( x_1^k s_1^k + (x_{2:n^k}^k)^T s_{2:n^k}^k \right) \\ &\geq \sum_{k=1}^{r} \left( x_1^k s_1^k - \left\| (x_{2:n^k}^k) \right\| \left\| s_{2:n^k}^k \right\| \right) \\ &\geq 0. \end{aligned}$$

## We can conclude

$$\begin{aligned} x_1^k s_1^k &= \left\| x_{2:n^k}^k \right\| \left\| s_{2:n^k}^k \right\|, \\ x_1^k &= \left\| x_{2:n^k}^k \right\| \\ s_1^k &= \left\| s_{2:n^k}^k \right\|. \end{aligned}$$

(Why?).

Now

$$\left|(x^k_{2:n^k})^T s^k_{2:n^k}\right| = \left\|x^k_{2:n^k}\right\| \left\|s^k_{2:n^k}\right\|$$

can only be the case if

$$\exists \alpha : \; x^k_{2:n^k} = \alpha s^k_{2:n^k}.$$

Therefore,

$$\begin{array}{rcl} 0 & = & (x^k)^T s^k \\ & = & x^k_1 s^k_1 + \alpha \left\|s^k_{2:n^k}\right\|^2 \\ & = & x^k_1 s^k_1 + \alpha (s^k_1)^2 \end{array}$$

and

$$\alpha = -\frac{x^k_1}{s^k_1}$$

implying that the complementarity conditions $X^k s^k = 0$ are satisfied.

# Optimality conditions

**Theorem 2** *Assume that* $\exists x, s \in \text{int}(\mathcal{K})$ *such that* $Ax = b$ *and* $A^T y + s = c$ *for some* $y$ *then* $(x, y, s)$ *is an optimal solution if and only if*

$$\begin{array}{rcll} Ax & = & b, & x \in \mathcal{K}, \\ A^T y + s & = & c, & s \in \mathcal{K}, \\ X^k s^k & = & 0, & k = 1, \ldots, r. \end{array}$$

# Further facts about the dual cone

.

Recall given a convex cone $\mathcal{K}$ then the dual cone $\mathcal{K}^*$ is given by

$$\mathcal{K}^* := \{s : \ s^T x \geq 0, \ \forall x \in \mathcal{K}\}.$$

## Lemma 5

1. *If $\mathcal{K}$ is convex and closed, then $(\mathcal{K}^*)^* = \mathcal{K}$.*
2. *$\mathcal{K}^*$ is closed and convex. (Holds even if $\mathcal{K}$ is not convex but is a cone).*
3. *$\mathcal{K}_1 \subseteq \mathcal{K}_2$ implies $\mathcal{K}_2^* \subseteq \mathcal{K}_1^*$.*

# Other cones

.

**Lemma 6** *Let*

$$\mathcal{K} = \{x \in \mathcal{R}^n : \ x_1 \geq \|x_{2:n}\|_1\}$$

*then*

$$\mathcal{K}^* = \{x \in \mathcal{R}^n : \ x_1 \geq \|x_{2:n}\|_\infty\}.$$

# Nonlinear cones



More generally we have

**Lemma 7** *Let $\|\|$ be a norm on $\mathcal{R}^n$ and then define*

$$\mathcal{K} = \{x \in \mathcal{R}^n : \ x_1 \geq \|x_{2:n}\|\}$$

*then*

$$\mathcal{K}^* = \{x \in \mathcal{R}^n : \ x_1 \geq \|x_{2:n}\|_*\}.$$

*where $\|\|_*$ is the dual norm i.e.*

$$\|u\|_* = \sup\{u^T x : \|x\| \leq 1\}.$$

# Summary

- Conic duality has been studied.
- Shown pathological duality cases exists.
- Derived the complementarity conditions for conic quadratic optimization problems.
- [2] is good reference.

# Exercises

**Exercise 1** *Prove that the quadratic cone is self-dual.*

**Exercise 2** *What is the dual problem to*

$$\begin{array}{lrcll} \max & b^T y & & & \\ \textit{st} & (A^k)^T y + s^k & = & c^k, & k = 1, \ldots, r, \\ & s^k \in (\mathcal{K}^k)^*, & & & k = 1, \ldots, r. \end{array}$$

*where* $\mathcal{K}^k$ *is a quadratic cone.*

## Exercise 3

*Show that the dual problem corresponding to*

$$
\begin{array}{lllll}
\min & f^T x & & & \\
\textit{st} & ||A^i x - b^i|| & \leq & c_{i:} x - d_i, & i = 1, \ldots, k, \\
 & Hx & = & h & 
\end{array}
$$

*is*

$$
\begin{array}{lllll}
\max & b^T z + d^T w + h^T v & & & \\
\textit{st} & A^T z + C^T w + H^T v & = & f, & \\
 & ||z^i|| & \leq & w_i, & i = 1, \ldots, k,
\end{array}
$$

**Exercise 4** *Prove Lemma 6.*

# References



[1] A. Ben-Tal and A Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS/SIAM Series on Optimization. SIAM, 2001.

[2] S.P. Boyd, S.J. Kim, L. Vandenberghe, and A. Hassibi. A Tutorial on Geometric Programming. Technical report, ISL, Electrical Engineering Department, Stanford University, Stanford, CA, 2004. Available at `http://www.stanford.edu/~boyd/gp_tutorial.html`.

[3] J. Renegar. *A mathematical view of interior-point methods in convex optimization*. MPS/SIAM Series on Optimization. SIAM, 2001.