# Conic quadratic optimization - part 3

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# Introduction

# Topics



- Important applications of conic quadratic optimization.
  - Robust linear optimization.
  - Portfolio optimization.

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# Robust optimization

# Robust optimization

- A very important application of conic quadratic optimization is robust optimization.
- Robust optimization assumes the problem data e.g. $A$ is not known exactly.
- Tries to compute a robust solution.

# A motivating example



Consider the toy linear optimization problem:

A company produces two kinds of drugs, DrugI and DrugII, containing a specific active agent A, which is extracted from raw materials which should be purchased on the market. The drug production data are as follows:

| Drug | Selling price, $ per 1000 packs | Content of agent A, g per 1000 packs |
|---|---|---|
| DrugI | 6,200 | 0.500 |
| DrugII | 6,900 | 0.600 |

| Drug | Production expenses per 1000 packs | | |
|---|---|---|---|
| | manpower, hours | equipment, hours | operational costs, $ |
| DrugI | 90.0 | 40.0 | 700 |
| DrugII | 100.0 | 50.0 | 800 |

There are two kinds of raw materials, RawI and RawII, which can be used as sources of the active agent. The related data are as follows:

| Raw material | Purchasing price, \$ per kg | Content of agent A, g per kg |
|---|---|---|
| RawI | 100.00 | 0.01 |
| RawII | 199.90 | 0.02 |

Finally, the per month resources dedicated to producing the drugs are as follows:

| Budget, \$ | Manpower, hours | Equipment, hours | Capacity of raw materials storage, kg |
|---|---|---|---|
| 100,000 | 2,000 | 800 | 1,000 |

The problem is *to find the production plan which maximizes the profit of the company*.

The problem can be immediately posed as the following linear programming program:
maximize

$$
\begin{aligned}
&-(100 \cdot \texttt{RawI} + 199.90 \cdot \texttt{RawII} + 700 \cdot \texttt{DrugI} + 800 \cdot \texttt{DrugII}\ )\ \ \text{(cos}\\
&\quad + (6200 \cdot \texttt{DrugI} + 6900 \cdot \texttt{DrugII})\ \ \text{(income)}
\end{aligned}
$$

subject to

$$
\begin{array}{rcl}
0.01 \cdot \texttt{RawI} + 0.02 \cdot \texttt{RawII} - 0.500 \cdot \texttt{DrugI} - 0.600 \cdot \texttt{DrugII} & \geq & 0 \\
\texttt{RawI} + \texttt{RawII} & \leq & 1000 \\
90.0 \cdot \texttt{DrugI} + 100.0 \cdot \texttt{DrugII} & \leq & 2000 \\
40.0 \cdot \texttt{DrugI} + 50.0 \cdot \texttt{DrugII} & \leq & 800 \\
100.0 \cdot \texttt{RawI} + 199.90 \cdot \texttt{RawII} + 700 \cdot \texttt{DrugI} + 800 \cdot \texttt{DrugII} & \leq & 100000 \\
\texttt{RawI}, \texttt{RawII}, \texttt{DrugI}, \texttt{DrugII} & \geq & 0
\end{array}
$$

Explanation of constraints:

- balance of active agent
- storage restriction
- manpower restriction
- equipment restriction
- budget restriction

The optimal solution:

```
*** Optimal value: 8819.658
*** Optimal solution:
RawI:      0.000
RawII:   438.789
DrugI:    17.552
DrugII:    0.000
```

Comments

- The company makes a profit $8819$ on a budget of 100,000 i.e. 8.8%.
- The balance constraint is active as could have been guessed.
- Is there anything wrong with the solution?

- Is it likely that `RawI` contains exactly 0.01 g per kg of agent A?

- Is it likely that `RawI` contains exactly 0.01 g per kg of agent A? **No.**
- Reasonable assumption: The contents of the active agent $a_I$ in `RawI` and $a_{II}$ in `RawII` in the raw materials are random variables.
- Assume instead:

$$a_I = \begin{cases} 0.0095, & p = 0.5 \\ 0.0105, & p = 0.5 \end{cases}$$

and

$$a_{II} = \begin{cases} 0.0196, & p = 0.5 \\ 0.0204, & p = 0.5 \end{cases}$$

where $p$ is a probability.

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- The expected profit is $7854$.
- Conclusion:
  We see that in our toy example *pretty small (and unavoidable in reality) perturbations of the data may make the optimal solution infeasible, and a straightforward adjustment to the actual solution values may heavily affect the solution quality.*

# Robust linear optimization



The standard linear optimization problem:

$$\begin{array}{lll} \min & c^T x & \\ \text{st} & a_{i:} x & \leq & b_i, & \forall i. \end{array}$$

Assume:

$$a_{i:}^T \in \mathcal{E}_i := \{z : \ z = \bar{a}_{i:}^T + H^i y, \ \ \|y\| \leq 1\},$$

where

$$H^i \in \mathbf{R}^{n \times l_i}.$$

Observe:

- $\mathcal{E}_i$ is an ellipsoid.

For a fixed $x$ we have

$$\begin{aligned}
\max_{a_{i:}\in\mathcal{E}_i} a_{i:}x &= \max_{\|y\|\le 1} x^T(\bar{a}_{i:}^T + H^i y) \\
&= \bar{a}_{i:}x + \max_{\|y\|\le 1} x^T H^i y \\
&= \bar{a}_{i:}x + \left\|(H^i)^T x\right\|.
\end{aligned}$$

(Why does the last equality holds?)

Therefore,

$$\begin{array}{lllll} \min & c^T x & & & \\ \text{st} & a_{i:} x & \leq & b_i, & a_{i:}^T \in \mathcal{E}_i, \quad \forall i \end{array}$$

and

$$\begin{array}{lllll} \min & c^T x & & & \\ \text{st} & \bar{a}_{i:} x + \left\| (H^i)^T x \right\| & \leq & b_i, & \forall i \end{array}$$

are equivalent.

# A simple example



Consider

$$\begin{array}{lrcl} \min & -x_1 & & \\ \text{st} & a_1 x_1 + a_2 x_2 & \leq & 1, \\ & x_1, x_2 \geq 0. & & \end{array}$$

Assuming that $a_1 = a_2 = 1$ then the optimal solution is

$$x_1 = 1 \text{ and } x_2 = 0.$$

Notes

- The optimal solution is on the boundary (holds generically).
- The optimal solution is infeasible if $a_1 > 1$ and therefore not robust.

Next consider the robust version

$$\begin{array}{lrcl} \min & -x_1 & & \\ \text{st} & a_1 x_1 + a_2 x_2 & \leq & 1, \quad \forall (a_1, a_2) \in \mathcal{E} \\ & x_1, x_2 \geq 0. & & \end{array}$$

where

$$\mathcal{E} := \{(a_1, a_2) : \ (a_1, a_2) = (1,1) + \theta y, \ \ \|y\| \leq 1\}$$

and $\theta$ is a fixed nonnegative number. Equivalent robust version

$$\begin{array}{lrcl} \min & -x_1 & & \\ \text{st} & x_1 + x_2 + \theta \left\|(x_1, x_2)\right\| & \leq & 1, \\ & x_1, x_2 \geq 0. & & \end{array}$$

## Notes

- The optimal solution is $(x_1, x_2) = \left(\frac{1}{1+\theta}, 0\right)$.
- The optimal solution is in the interior of

$$\{(x_1, x_2) : \; x_1 + x_2 \leq 1\}$$

  for $\theta > 0$.
- The robust version push the optimal solution into the interior of the original feasible region.
- Therefore, the optimal solution is still feasible even for small changes in the problem data.
- Clearly, there is a **tradeoff** between "robustness" and the objective value.

# A statistical interpretation



Assumptions:

- $a_{i:}$ are independent Gaussian random vectors i.e.

$$a_{i:} \sim N(\bar{a}_{i:}, \Sigma_i).$$

Problem:

$$\begin{array}{lll} \min & c^T x & \\ \text{st} & \text{Prob}(a_{i:}x \leq b_i) & \geq p, \forall i. \end{array}$$

Now

$$\text{Prob}(a_{i:}x \leq b_i) \geq p$$

is equivalent to

$$\text{Prob}\left(\frac{a_{i:}x - \mu}{\sigma_i} \leq \frac{b_i - \mu}{\sigma_i}\right) \geq p$$

where

$$\mu = \bar{a}_{:i}x \text{ and } \sigma_i = \left\| \Sigma^{\frac{1}{2}} x \right\|.$$

Clearly

$$\frac{a_{i:}x - \bar{a}_{:i}x}{\left\| \Sigma_i^{\frac{1}{2}} x \right\|} \sim N(0,1).$$

Hence,

$$\frac{b_i - \bar{a}_{:i}x}{\left\| \Sigma_i^{\frac{1}{2}} x \right\|} \geq \Phi^{-1}(p)$$

where

$$\Phi(z) := \frac{1}{2\pi} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

Thus

$$b_i \geq \bar{a}_{:i}x + \Phi^{-1}(p) \left\| \Sigma_i^{\frac{1}{2}} x \right\|.$$

Equivalent problem:

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{st} & \bar{a}_{i:}x + \Phi^{-1}(p)\left\|\Sigma_i^{1/2}x\right\| & \leq & b_i, \forall i. \end{array}$$

Notes:

- For $p \geq 0.5$ then $\Phi^{-1}(p) \geq 0$.
- Hence, is a conic quadratic problem for $p \geq 0.5$.
- Is called *chance constrained* optimization.

# Portfolio optimization

# The problem



- Select a portfolio of assets i.e. stocks, bonds, etc.
- Such that a large return with a low risk is obtained.
- Assumptions:
  - An initial portfolio is available.
  - A single period.
  - One of the assets is risk free i.e. cash.

# Formal definition



Parameters:

- A portfolio can consist of $n$ traded assets numbered $1, 2, \ldots$ held over a period of time
- $w_j^0$ is the initial holding of asset $j$ where $\sum_j w_j^0 > 0$.
- $r_j$ is the return on asset $j$ assumed to be a random variable. $r$ has a known mean $\bar{r}$ and covariance $\Sigma$.

Variables:

- $x_j$ is the amount of asset $j$ traded.
  - If $x_j > 0$, then the amount of asset $j$ is increased (by purchasing).
  - If $x_j < 0$, then the amount of asset $j$ is decreased (by selling).

# Tradeoff



Observe

- Return (expected return)

$$E[r^T(w^0 + x)] = \bar{r}^T(w^0 + x)$$

- Risk (variance)

$$V[r^T(w^0 + x)] = (w^0 + x)^T \Sigma (w^0 + x)$$

- High return and a small risk i.e. small variance is desired.
- There is a trade-off between return and risk.

- Expected return and variance can be nontrivial to estimate.
- By definition $\Sigma$ is positive semi-definite and

  $$\begin{aligned} \text{Std. dev.} &= \left\| \Sigma^{\frac{1}{2}}(w^0 + x) \right\| \\ &= \left\| L^T(w^0 + x) \right\| \end{aligned}$$

  where $L$ is **any** matrix such that

  $$\Sigma = LL^T$$

  i.e. for instance the Cholesky factor.
- A low rank of $\Sigma$ is advantageous from a computational point of view.

First model:

$$\begin{array}{lrcl} \min & (w^0 + x)^T \Sigma (w^0 + x) & & \\ \text{st} & \bar{r}^T (w^0 + x) & = & t, \\ & e^T x & = & 0, \end{array}$$

where $e := (1, \ldots, 1)^T$.

Model:

- Minimizes the variance.
- While selecting a portfolio having an expected target return of $t$.
- Satisfying the budget or self-financing constraint.

Usage:

- Solved for different values of $t$.
- Investor choose the portfolio that according to his/her preferences has the best relation between risk and return.

Comments:

- Is a convex quadratic optimization problem.
- Can be formulated as conic a quadratic optimization problem.

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Conic quadratic reformulation:

$$
\begin{array}{lrcl}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
& \bar{r}^T(w^0+x) & = & t, \\
& e^T x & = & 0, \\
& f \geq \|g\| \,. & &
\end{array}
$$

- Minimizes the standard deviation instead of the variance.
- Is a conic quadratic optimization problem.
- $\Sigma^{\frac{1}{2}}$ can be replaced by any matrix $L$ where $\Sigma = LL^T$. A low rank $L$ is computationally advantageous.
- If linear inequality constraints are added to the problem it gets harder.

## Alternative formulation

$$
\begin{array}{lrcl}
\max & \bar{r}^T(w^0+x) & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
& f & = & \hat{f}, \\
& e^T x & = & 0, \\
& f \geq \|g\|. & &
\end{array}
$$

- Maximizes the expected return.
- While making sure the standard deviation is bounded by $\hat{f}$ chosen by the investor.

## Short selling



- Short selling is allowed i.e.

$$w_j^0 + x_j < 0$$

  is allowed.
- How does short selling works?
  - Borrow the asset from someone now and then sell it.
  - At the end of the period buy the asset back.
  - Return it to the lender with the return.
  - You make money if prices decrease and loose if they increase.
- What is the potential loss of short selling?

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  - At the end of the period buy the asset back.
  - Return it to the lender with the return.
  - You make money if prices decrease and loose if they increase.
- What is the potential loss of short selling?
  - Infinite.

New model

$$
\begin{array}{lrcl}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0 + x) - g & = & 0, \\
 & \bar{r}^T(w^0 + x) & = & t, \\
 & e^T x & = & 0, \\
 & w_j^0 + x_j & \geq & 0, \\
 & f \geq \|g\| \,. & &
\end{array}
$$

- Eliminates short selling completely by adding the constraint

$$w_j^0 + x_j \geq 0.$$

Alternatives:

- Allow short selling of $s_j$ for asset $j$

$$w_j^0 + x_j \geq -s_j.$$

- Limits the total of the short positions to a fraction $\gamma$ of the total of the long positions.

$$\begin{array}{rcl} \sum_j (w_j + x_j) & = & h^+ - h^-, \\ \sum_j h_j^- & \leq & \gamma \sum_j h_j^+, \\ h^+, h^- & \geq & 0, \end{array}$$

where $h_j^+$ and $h_j^-$ are variables.

## Diversification constraints



- Limit the amount invested in each asset to $b_j$:

$$w_j^0 + x_j \leq b_j.$$

- Limit the relative amount invested in each asset to $\gamma_j$:

$$w_j^0 + x_j \leq \gamma_j \sum_j (w_j^0 + x_j).$$

- Limit the relative amount invested in a group of assets ($\mathcal{J}$) by $\gamma$:

$$\sum_{j \in \mathcal{J}} (w_j^0 + x_j) \leq \gamma \sum_j (w_j^0 + x_j).$$

# Transaction costs

- We have assumed no transactions cost i.e the cost of trading is 0.
- According to [3] trans. cost has the form

$$\text{commission} + \frac{\text{bid}}{\text{ask}} - \text{spread} + \theta\sqrt{\frac{\text{trade volume}}{\text{daily volume}}}.$$

- The market impact cost

$$\theta\sqrt{\frac{\text{trade volume}}{\text{daily volume}}}$$

  can be most significant term.
- $\theta$ has to be estimated.
- “daily volume” may be hard to know.

# Linear transaction costs



- Assume transaction cost can be approximated by a (piecewise) linear function e.g.

$$T(x_i) = \begin{cases} -c_j^- x_j, & x_j \leq 0, \\ c_j^+ x_j, & x_j > 0. \end{cases}$$

- An even more realistic assumption is linear transactions cost plus a fixed trading cost i.e.

$$T(x_i) = \begin{cases} b_j - c_j^- x_j, & x_j < 0, \\ 0, & x_j = 0, \\ b_j + c_j^+ x_j, & x_j > 0. \end{cases}$$

# No fixed transaction costs



A model including linear transaction costs:

$$
\begin{array}{lrcl}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
& \bar{r}^T(w^0+x) & = & t, \\
& e^Tx+e^Ty & = & 0, \\
& c_j^+x_j & \leq & y_j, \\
& -c_j^-x_j & \leq & y_j, \\
& f \geq \|g\|. & &
\end{array}
$$

- $y_j$ is the transaction cost associated with asset $j$.
- We should prove that one of the inequalities

$$c_j^+x_j \leq y_j \text{ and } -c_j^-x_j \leq y_j$$

hold as equality at optimum. Not true unfortunately!

# A patch



Assuming a risk free asset with positive return then the model

$$
\begin{array}{lrcl}
\max & \bar{r}^T(w^0+x) & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
 & f & = & \hat{f}, \\
 & e^T x + e^T y & = & 0, \\
 & c_j^+ x_j & \leq & y_j, \\
 & -c_j^- x_j & \leq & y_j, \\
 & f \geq \|g\| . & &
\end{array}
$$

does not have the problem.

# The Mitchell and Braun model



Mitchell and Braun [2] suggests to minimize

$$\frac{\Sigma^{\frac{1}{2}}(w^0 + x)}{e^T w^0 - e^T y} = \frac{\text{std. dev.}}{\text{invested amount}}$$

i.e. minimize the standard deviation per invested \$. Seems quit reasonable.

## Model

$$
\begin{array}{lrcl}
\min & \frac{f}{v} & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0 + x) - g & = & 0, \\
& \bar{r}^T(w^0 + x) & = & t, \\
& e^T x + e^T y & = & 0, \\
& c_j^+ x_j & \leq & y_j, \\
& -c_j^- x_j & \leq & y_j, \\
& e^T w^0 - e^T y & = & v, \\
& v & > & 0, \\
& f \geq \|g\| \,. & &
\end{array}
$$

- $v$ is variable and is the invested amount.

Define

$$x = \frac{\hat{x}}{\tau}$$

and similarly for all the other variables. Also multiply each constraint by $\tau$. New model:

$$\begin{array}{lrcl}
\min & \frac{\hat{f}}{\hat{v}} & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0\tau + \hat{x}) - \hat{g} & = & 0, \\
 & \bar{r}^T(w^0\tau + \hat{x}) & = & t\tau, \\
 & e^T\hat{x} + e^T\hat{y} & = & 0, \\
 & c_j^+\hat{x}_j & \leq & \hat{y}_j, \\
 & -c_j^-\hat{x}_j & \leq & \hat{y}_j, \\
 & e^Tw^0\tau - e^T\hat{y} & = & \hat{v}, \\
 & \hat{v}, \tau & > & 0, \\
 & \hat{f} \geq \left\|\hat{g}\right\|. & &
\end{array}$$

- Is homogeneous.
- Has a linear fractional objective.
- Therefore, we can arbitrarily fix $v$ to 1.

Final model:

$$
\begin{array}{lrcl}
\min & \hat{f} & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0\tau + \hat{x}) - \hat{g} & = & 0, \\
& \bar{r}^T(w^0\tau + \hat{x}) & = & t\tau, \\
& e^T\hat{x} + e^T\hat{y} & = & 0, \\
& c_j^+\hat{x}_j & \leq & \hat{y}_j, \\
& -c_j^-\hat{x}_j & \leq & \hat{y}_j, \\
& e^T w^0\tau - e^T\hat{y} & = & \hat{v}, \\
& \hat{v} & = & 1, \\
& \tau & > & 0, \\
& f \geq \|g\| \,. & &
\end{array}
$$

- Is a conic quadratic optimization problem.

# Fixed trading costs



A model including linear transaction costs:

$$
\begin{array}{lccc}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
 & \bar{r}^T(w^0+x) & = & t, \\
 & e^T x + e^T y & = & 0, \\
 & b_j z_j + c_j^+ x_j & \leq & y_j, \\
 & b_j z_j - c_j^- x_j & \leq & y_j, \\
 & x_j & \leq & u_j z_j, \\
 & -x_j & \leq & l_j z_j, \\
 & z_j \in \{0,1\} & & \\
 & f \geq \|g\| \,. & &
\end{array}
$$

- $l_j$ and $u_j$ are known parameters such that

$$l_j \leq x_j \leq u_j$$

- In theory a very hard problem. Tight bounds $l_j$ and $u_j$ help.
- Various heuristics can be designed. See [1].
- Or branch and bound can be used.

# The market impact term

- If you sell (buy) a lot of asset the price is likely to go down (up).
- This is captured by the market impact term

$$\theta\sqrt{\frac{\text{trade volume}}{\text{daily volume}}} \approx m_j\sqrt{x_j}.$$

$m_j$ has to be estimated.

Market impact cost can be included as follows

$$
\begin{array}{lccc}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
 & \bar{r}^T(w^0+x) & = & t, \\
 & e^Tx+e^Ty & = & 0, \\
 & |x_j|(m_j|x_j|^{\frac{1}{2}}) & \leq & y_j, \\
 & f \geq \|g\|. & &
\end{array}
$$

Now define the variable transformation

$$y_j = m_j \bar{y}_j$$

then we obtain

$$\begin{array}{lrcl}
\min & f & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0 + x) - g & = & 0, \\
& \bar{r}^T(w^0 + x) & = & t, \\
& e^T x + m^T \bar{y} & = & 0, \\
& |x_j|^{3/2} & \leq & \bar{y}_j, \\
& f \geq \|g\|. & &
\end{array}$$

The set

$$|x_j|^{3/2} \leq \bar{y}_j$$

can be modeled by

$$\begin{array}{rcl} x_j & \leq & z_j, \\ -x_j & \leq & z_j, \\ z_j^2 & \leq & 2 s_j \bar{y}_j, \\ u_j^2 & \leq & 2 v_j q_j, \\ z_j & = & v_j, \\ s_j & = & u_j, \\ q_j & = & \frac{1}{8}, \\ q_j, s_j, \bar{y}_j, v_j, q_j & \geq & 0. \end{array}$$

# Shortfall risk



- Assume
$$r \sim N(\bar{r}, \Sigma).$$
- May not be a totally reasonable assumption.
- Shortfall risk constraint
$$Prob(W \geq W^{low}) \geq \eta$$
where $\eta \geq 0.5$ and $W^{low} > 0$. Both chosen by the investor.
- I.e. the wealth should be greater than a lower threshold with a high probability.

Now

$$Prob(W \geq W^{low}) \geq \eta$$

implies

$$Prob\left(\frac{W-\mu}{\sigma} \geq \frac{W^{low}-\mu}{\sigma}\right) \leq 1-\eta$$

where

$$\mu = \bar{r}(w^0+x) \text{ and } \sigma = \left\|\Sigma^{\frac{1}{2}}(w^0+x)\right\|$$

Now

$$z = \frac{W-\mu}{\sigma} \sim N(0,1).$$

Given

$$\Phi(z) := \frac{1}{2\pi} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

the shortfall risk constraint is equivalent to

$$\Phi^{-1}(\eta) \leq \frac{\bar{r}^T (w^0 + x) - W^{low}}{\left\| \Sigma^{\frac{1}{2}} (w^0 + x) \right\|}$$

or

$$\Phi^{-1}(\eta) \left\| \Sigma^{\frac{1}{2}} (w^0 + x) \right\| \leq \bar{r}^T (w^0 + x) - W^{low}.$$

- Is a conic quadratic constraint for $\eta \geq 0$.

# Example model

A model with a single shortfall risk constraint:

$$\begin{array}{lrcl}
\max & \bar{r}^T(w^0+x) & & \\
\text{st} & \Sigma^{\frac{1}{2}}(w^0+x)-g & = & 0, \\
 & e^T x & = & 0, \\
 & \frac{1}{\Phi^{-1}(\eta)}(\bar{r}^T(w^0+x)-W^{low}) & \geq & f \\
 & f \geq \|g\| . & &
\end{array}$$

- Multiple shortfall constraints are possible.

# References portfolio optimization



- [3] deals with portfolio optimization.
- [1] contains a lot the material presented here.

# Summary

- Robust optimization is an important application for conic quadratic optimization.
  - Can be given a nice chance constrained interpretation.
- The Markowitz portfolio model and its variants can be formulated as quadratic optimization problem.
  - Probably the most commercially most important application of conic quadratic optimization as of 2006.

# References



[1] M. S. Lobo, M. Fazel, and S. Boyd. Portfolio optimization with linear and fixed transaction costs. Technical report, 2005. To appear in Annals of Operations Research. http://www.cds.caltech.edu/~maryam/portfolio.html.

[2] John E. Micthell and Stephen Braun. Reblancing an investment portfolio in the precense of convex transaction costs. Technical report, Department of Mathematical Sciences, Rensselaer Polytechnic Insritue, Troy, NY, 12180, December 2004.

[3] Bernd Scherer. *Portfolio construction and risk budgeting*. Risk Books, 2 edition, 2004.