Improving an interior-point algorithm for multicommodity flows by quadratic regularizations

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Improving an interior-point algorithm for multicommodity flows by quadratic regularizations [*]

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Abstract

One of the best approaches for some classes of multicommodity flow problems is a specialized interior-point method that solves the normal equations by a combination of Cholesky factorizations and preconditioned conjugate gradient. Its efficiency depends on the spectral radius—in [0,1)—of a certain matrix in the definition of the preconditioner. In a recent work the authors improved this algorithm (i.e., reduced the spectral radius) for general block-angular problems by adding a quadratic regularization to the logarithmic barrier. This barrier was shown to be self-concordant, which guarantees the convergence and polynomial complexity of the algorithm. In this work we focus on linear multicommodity problems, a particular case of primal block-angular ones. General results are tailored for multicommodity flows, allowing a local sensitivity analysis on the effect of the regularization. Extensive computational results on some standard and some difficult instances, testing several regularization strategies, are also provided. These results show that the regularized interior-point algorithm is more efficient than the nonregularized one. From this work it can be concluded that, if interior-point methods based on conjugate gradients are used, linear multicommodity flow problems are most efficiently solved as a sequence of quadratic ones.

Key words: interior-point methods, multicommodity network flows, precondi-

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tioned conjugate gradient, regularizations, large-scale computational optimization

1 Introduction

Multicommodity flows are widely used as a modeling tool in many fields as, e.g., in telecommunications and transportation problems. This kind of models are usually very large linear programming problems, and some difficult instances have shown to be challenging for state-of-the-art solvers [8]. For these difficult instances, the specialized interior-point algorithm of [7] can be a competitive option. In this work that approach is improved by adding a quadratic regularization. In particular, as it will be shown, the quality of the preconditioner of the PCG solver used by the algorithm is improved by the regularization. The resulting multicommodity flow code is more efficient than the original nonregularized one of [7]. The new multicommodity algorithm relies on theoretical results developed in [11] for a more general class of problems.

In the last two decades there has been a significant amount of research in the field of multicommodity flows, mainly for linear problems. Some of the solution strategies can be broadly classified into four main categories: simplex-based methods [13, 20], decomposition methods [4, 15, 17], approximation methods [5], and interior-point methods [4, 7, 17]. Some of the approaches for linear multicommodity flows were compared in [14]. Significant advances have also been made for nonlinear multicommodity flows. Among them we find active set methods [13], ACCPM approaches [3, 17], interior-point methods for quadratic problems [9], proximal point algorithms [23], and bundle-type decomposition [19]. A description and empirical evaluation of additional nonlinear multicommodity algorithms can be found in the survey [24].

The specialized interior-point algorithm for multicommodity flows extended in this work was first suggested in [7]. Given a directed network of n' arcs and $m' + 1$ nodes, the algorithm considers this general formulation for multicommodity flows:

$$\min \quad \sum_{i=0}^{k} ({c^i}^T x^i + {x^i}^T Q_i x^i) \tag{1a}$$

$$\text{subject to} \quad \begin{bmatrix} N & & & & 0 \\ & N & & & 0 \\ & & \ddots & & \vdots \\ & & & N & 0 \\ I & I & \dots & I & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \\ x^0 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^k \\ u \end{bmatrix} \tag{1b}$$

$$0 \le x^i \le u^i \qquad i = 0, \dots, k. \tag{1c}$$

Vectors $x^i \in \mathbb{R}^{n'}$, $i = 1, \dots, k$, are the flows for commodity i, while $x^0 \in \mathbb{R}^{n'}$ are the slacks of capacity constraints. The node-arc incidence matrix of the directed graph is $N \in \mathbb{R}^{m' \times n'}$. We assume N has full row-rank, which can always be achieved by removing one of the redundant constraints associated to some node. Note that N may represent any directed graph, thus the algorithm is not tailored to some particular network and it can deal with any multicommodity flow problem. The identity matrix, used in the formulation of linking constraints, is represented by I. The arc capacities for all the commodities are $u \in \mathbb{R}^{n'}$, while $u^i \in \mathbb{R}^{n'}, i = 1, \ldots, n'$, are the individual capacities per commodity; $u^0 \in \mathbb{R}^{n'}$ are the upper bounds of slacks x^0, and in general we have $u^0 = u$. Vectors $b^i \in \mathbb{R}^{m'}, i = 1, \ldots, k$, are the node supply/demands for each commodity. Vectors $c^i \in \mathbb{R}^{n'}, i = 1, \ldots, k$, are the arc linear costs per commodity, and the diagonal positive semidefinite matrices $Q_i \in \mathbb{R}^{n' \times n'}, i = 1, \ldots, k$, denote the arc quadratic costs. Note that the algorithm can also deal with linear costs $c^0 \in \mathbb{R}^{n'}$ and quadratic costs $Q_0 \in \mathbb{R}^{n' \times n'}$ (Q_0 diagonal and positive semidefinite) for slacks; this can be useful for problems that involve quadratic costs for the total flow on arcs, since $\sum_{i=1}^{k} x^i = u - x^0$. Clearly, for linear multicommodity problems $Q_i = 0$. However, the regularized algorithm will make use of this quadratic term.

The structure of this paper is as follows. Section 2 outlines the specialized interior-point algorithm for primal block-angular problems, provides the main theoretical results about the improvement due to a quadratic term, and describes the regularized variant of the specialized algorithm and its main properties. Section 3 particularizes general results for primal block-angular problems to multicommodity flows. Using these particular results, Section 4 performs a sensitivity analysis to the addition of a quadratic regularization term. Section 5 evaluates several regularization strategies. Finally, Section 6 provides computational results with an implementation of the regularized algorithm.

2 Outline of the regularized interior-point algorithm for multicommodity flows

The specialized algorithm, initially developed for multicommodity flows [7], was extended for general primal block-angular problems in [10]. The improved regularized version [11] was developed for this more general formulation:

$$\min \quad \sum_{i=0}^{k} (c^{i^T} x^i + x^{i^T} Q_i x^i) \tag{2a}$$

$$\text{subject to} \quad \begin{bmatrix} N_1 & & & & 0 \\ & N_2 & & & 0 \\ & & \ddots & & \vdots \\ & & & N_k & 0 \\ L_1 & L_2 & \ldots & L_k & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \\ x^0 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^k \\ b^0 \end{bmatrix} \tag{2b}$$

$$0 \le x^i \le u^i \qquad i = 0, \ldots, k. \tag{2c}$$

The main difference between (2) and (1) is that matrices $N_i \in \mathbb{R}^{m_i \times n_i}$ and $L_i \in \mathbb{R}^{l \times n_i}$ may have any structure, and be of different dimensions for each $i = 1, \ldots, k$, l being the number of linking constraints. For some particular matrices N_i and L_i it is possible to tailor general results of below Subsection 2.2 and Section 3. We also restrict our considerations to the separable case where $Q_i \in \mathbb{R}^{n_i \times n_i}$, $i = 0, \ldots, k$, are diagonal positive semidefinite matrices.

2.1 The specialized algorithm

Problem (2) can be written as

$$\begin{array}{rl} \min & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array} \tag{3}$$

where $c, x, u \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $Q \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^m$. Note that $n = \tilde{n} + l$ and $m = \tilde{m} + l$, where $\tilde{n} = \sum_{i=1}^k n_i$ and $\tilde{m} = \sum_{i=1}^k m_i$; for the particular case of multicommodity problems (1), $\tilde{n} = kn'$, $\tilde{m} = km'$ and $l = n'$, and thus $n = (k+1)n'$ and $m = km' + n'$. Replacing inequalities in (3) by a logarithmic barrier with parameter $\mu > 0$ we obtain the logarithmic barrier problem

$$\begin{array}{rl} \min & B(x,\mu) \triangleq c^T x + \dfrac{1}{2} x^T Q x + \mu \left(-\displaystyle\sum_{i=1}^n \ln x_i - \sum_{i=1}^n \ln(u_i - x_i) \right) \\ \text{subject to} & Ax = b. \end{array} \tag{4}$$

The KKT conditions of (4) are [27]:

$$\begin{aligned} Ax &= b, && \text{(5a)} \\ A^T y - Qx + z - w &= c, && \text{(5b)} \\ XZe &= \mu e, && \text{(5c)} \\ (U - X)We &= \mu e, && \text{(5d)} \\ (z, w) &> 0 \quad u > x > 0. && \text{(5e)} \end{aligned}$$

Here, $e \in \mathbb{R}^n$ is a vector of 1's; $y \in \mathbb{R}^m$, $z, w \in \mathbb{R}^n$ are the Lagrange multipliers (or dual variables) of $Ax = b$, $x \geq 0$ and $x \leq u$, respectively; and matrices $X, Z, U, W \in \mathbb{R}^{n \times n}$ are diagonal matrices made up of vectors x, z, u, w. Equations (5a)–(5b) impose, respectively, primal and dual feasibility, while (5c)–(5d) impose complementarity. The normal equations for the Newton direction $(\Delta x, \Delta y, \Delta z)$ of (5) reduce to (see [10] for details)

$$\begin{aligned} (A\Theta A^T)\Delta y &= g && \text{(6)} \\ \Theta &= (Q + (U-X)^{-1}W + X^{-1}Z)^{-1}, && \text{(7)} \end{aligned}$$

for some right-hand-side g. For linear (i.e., $Q = 0$) or separable quadratic problems Θ is a diagonal positive definite matrix and it can be easily computed. Exploiting the structure of A and Θ in (2), the matrix of (6) can be written as

$$\begin{aligned} A\Theta A^T &= \left[\begin{array}{ccc|c} N_1\Theta_1 N_1^T & & & N_1\Theta_1 L_1^T \\ & \ddots & & \vdots \\ & & N_k\Theta_k N_k^T & N_k\Theta_k L_k^T \\ \hline L_1\Theta_1 N_1^T & \dots & L_k\Theta_k N_k^T & \Theta_0 + \sum_{i=1}^k L_i\Theta_i L_i^T \end{array} \right] \\ &= \left[\begin{array}{cc} B & C \\ C^T & D \end{array} \right], \end{aligned} \tag{8}$$

$B \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$, $C \in \mathbb{R}^{\tilde{m}\times l}$ and $D \in \mathbb{R}^{l\times l}$ being the blocks of $A\Theta A^T$, and Θ_i, $i = 0,\ldots,k$, the submatrices of Θ associated with the $k+1$ groups of variables in (2), i.e., $\Theta_i = (Q_i + (U_i - X_i)^{-1}W_i + X_i^{-1}Z_i)^{-1}$. Appropriately partitioning g and Δy in (6), the normal equations can be written as

$$\begin{bmatrix} B & C \\ C^T & D \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{9}$$

By eliminating Δy_1 from the first group of equations of (9), we obtain

$$(D - C^T B^{-1} C)\Delta y_2 = (g_2 - C^T B^{-1} g_1) \tag{10a}$$

$$B\Delta y_1 = (g_1 - C\Delta y_2). \tag{10b}$$

System (10b) is solved by a Cholesky factorization for each diagonal block $N_i\Theta_i N_i^T, i = 1\ldots k$, of B. This is a significant difference with interior-point approaches for single-commodity problems [16, 25], which solve $N_i\Theta_i N_i^T$ by a preconditioned conjugate gradient (PCG). The system with matrix $D - C^T B^{-1}C$, the Schur complement of (9), is solved by a PCG. The dimension of this system is l, which is the number of linking constraints. In [7] it was proved that the inverse of $(D - C^T B^{-1}C)$ can be computed as

$$(D - C^T B^{-1}C)^{-1} = \left(\sum_{i=0}^{\infty}(D^{-1}(C^T B^{-1}C))^i\right) D^{-1}. \tag{11}$$

The preconditioner M^{-1}, an approximation of $(D - C^T B^{-1}C)^{-1}$, is thus obtained by truncating the infinite power series (11) at some term h. In practice, $h = 0$ or $h = 1$ provide the best computational results.

2.2 Effect of the quadratic term

The effectiveness of the preconditioner depends on the spectral radius of matrix $D^{-1}(C^T B^{-1}C)$, which is always in $[0, 1)$ [7, Theorem 1]. The farther away from 1 is the spectral radius of $D^{-1}(C^T B^{-1}C)$, the better is the quality of the approximation of (11) obtained by truncation with $h = 0$ or $h = 1$. The next theorem and proposition from [11] show that the quadratic term in the objective function effectively reduces this spectral radius.

Theorem 1. *Let A be the constraint matrix of problem (2), with full row rank matrices $N_i \in \mathbb{R}^{m_i\times n_i}$ $i = 1,\ldots,k$, and at least one full row rank matrix $L_i \in \mathbb{R}^{l\times n_i}$, $i = 1,\ldots,k$. Let Θ be the diagonal positive definite matrix defined in (7), and $B \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$, $C \in \mathbb{R}^{\tilde{m}\times l}$ and $D \in \mathbb{R}^{l\times l}$ the submatrices of $A\Theta A^T$ defined in (8). Then, the spectral radius ρ of $D^{-1}(C^T B^{-1}C)$ is bounded by*

$$0 \le \rho \le \max_{j\in\{1,\ldots,l\}} \frac{\gamma_j}{\left(\frac{u_j}{v_j}\right)^2 \Theta_{0j} + \gamma_j} < 1, \tag{12}$$

where u is the eigenvector (or one of the eigenvectors) of $D^{-1}(C^T B^{-1}C)$ for ρ; γ_j, $j = 1,\ldots,l$, and $V = [V_1 \ldots V_l]$, are respectively the eigenvalues and matrix of columnwise eigenvectors of $\sum_{i=1}^{k} L_i\Theta_i {L_i}^T$; $v = V^T u$; and, abusing of notation, we assume that for $v_j = 0$, $(u_j/v_j)^2 = +\infty$.

Proposition 1. *Let assume the hypotheses of Theorem 1, and consider a linear problem and a quadratic one obtained by adding (likely small) quadratic costs $Q_i \succ 0$, $i = 1, \ldots, k$. Assume $\hat{u}_j/\hat{v}_j \leq u_j/v_j$, $j = 1, \ldots, l$, where "hatted" and "non-hatted" terms refer, respectively, to the linear and quadratic problems, and u and v are defined as in Theorem 1. Then bound (12) is smaller for the quadratic than for the linear problem.*

The technical assumption $\hat{u}_j/\hat{v}_j \leq u_j/v_j$, $j = 1, \ldots, l$ in Proposition 1 is needed to guarantee that the bound (12) is smaller if a quadratic term is added to a linear problem. The fulfillment of this assumption is problem dependent, and, for a general problem, it may not be easy to check. However, for the class of problems where $L_i, i = 1, \ldots, k$, are diagonal matrices, this assumption holds. This includes the class of multicommodity flow problems (see below Section 3 for details).

Preliminary computational results showed that the quadratic term in practice reduces the spectral radius, as predicted by the theory, and the overall number of PCG iterations and CPU time is significantly reduced. This explained the empirical results of previous works [9], where the specialized algorithm was more efficient for quadratic instances obtained from the linear ones by adding a separable quadratic convex cost than for the original linear instances.

2.3 The regularized algorithm

To reproduce the good behaviour of quadratic problems in linear ones a quadratic regularization term is added to the linear formulation (i.e., with $Q = 0$) of (3). Previously used regularized variants replaced $B(x, \mu)$ in (4) by a proximal point regularization

$$B_P(x, \mu) \triangleq c^T x + \frac{1}{2}(x - \bar{x})^T Q_P (x - \bar{x}) + \mu \left(-\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln(u_i - x_i) \right), \quad (13)$$

Q_P being a positive definite matrix and $\bar{x}$ the current point obtained by the interior-point algorithm. For instance, Q_P was the identity matrix in [26]; and Q_P was a diagonal matrix with small entries—dynamically updated at each interior-point iteration—in [2]. Unfortunately, these proximal point regularizations depend on the current point $\bar{x}$, and then they do not fit the general theory of structural optimization for interior-point methods [21]. In [11] the authors suggested the alternative regularized barrier problem

$$B_Q(x, \mu) \triangleq c^T x + \mu \left(\frac{1}{2} x^T Q x - \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln(u_i - x_i) \right), \quad (14)$$

Q being a diagonal positive semidefinite matrix. In this variant the regularization affects to the variables x (flows and slacks of (1)) instead to the directions as in (13). The regularized barrier function (14) was shown to be a self-concordant barrier [11] for upper-bounded problems and thus it fits the general interior-point theory of [21]. Since Q is diagonal, the self-concordant barrier

$$F_Q(x) = \frac{1}{2} x^T Q x - \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln(u_i - x_i)$$

of (14) can be written as a sum of self-concordant barriers for each component:

$$F_Q(x) = \sum_{i=1}^{n} F_{q_i}(x_i) = \sum_{i=1}^{n} \left(\frac{1}{2} q_i x_i^2 - \ln x_i - \ln(u_i - x_i) \right), \tag{15}$$

q_i being the diagonal entry of Q. The complexity of the interior-point algorithm in number of iterations is $O(\sqrt{\nu} \ln 1/\epsilon)$, where ϵ is the accuracy of the solution, and ν is the *parameter* of the self-concordant barrier of (14), which can be computed as $\nu = \sum_{i=1}^{n} \nu_i$, where ν_i is the parameter of the barrier $F_{q_i}(x_i)$ of (15) for component i (see [21] for details). In [11] the following result about ν_i was proved:

Proposition 2. *The parameter of the self-concordant barrier $F_{q_i}(x_i)$ of (15) in its domain $\{x_i : 0 < x_i < u_i\}$ is*

$$\begin{array}{lll} \nu_i = 1 & \textit{if} & 0 \le q_i \le 1/u_i^2, \\ \nu_i = q_i u_i^2 & \textit{if} & q_i \ge 1/u_i^2. \end{array} \tag{16}$$

The value $\nu_i = 1$ is the lowest possible one for any self-concordant barrier [21, Lemma 4.3.1], and therefore for small regularizations the regularized algorithm is in theory as efficient as the standard interior-point one. When $q_i \ge 1/u_i^2$ the complexity increases, but, as will be shown in next subsections, there is a wide range of values for which the number of interior-point iterations do not increase with the regularization term. Since the regularization term means less PCG iterations, the overall CPU time is reduced, making the regularized algorithm an effective approach for primal block-angular problems. It is also worth noting that the only (minor) change in the interior-point algorithm due to the regularized barrier problem (14) is that the dual feasibility condition (5b) is replaced by

$$A^T y - \mu Q x + z - w = c. \tag{17}$$

For linear problems, (17) and (5b) are equivalent when μ tends to zero. If the proximal point regularization (13) is used, the dual feasibility becomes

$$A^T y - Q_P (x - \bar{x}) + z - w = c. \tag{18}$$

Although, for linear problems, (18) is equal to (5b) when $x = \bar{x}$, the expression of Θ associated to (18) is

$$\Theta = (Q_P + (U - X)^{-1} W + X^{-1} Z)^{-1}, \tag{19}$$

while for (17) is

$$\Theta = (\mu Q + (U - X)^{-1} W + X^{-1} Z)^{-1}. \tag{20}$$

Note that when μ tends to zero (20) is a better approximation than (19) of the linear version of (7) (i.e., when $Q = 0$ in (7)). It has recently been proved [12] that the regularized central path defined by (5a), (17), (5c)–(5e) is well defined, i.e., it exists, it is unique, and it converges to a strictly complementary primal-dual solution.

3 The case of multicommodity flow problems

The bound provided by Theorem 1 is difficult to compute for general primal block-angular problems. However, for the particular case of multicommodity flow problems it reduces to a simple and computable form. Indeed, since $l = n'$, $N_i = N$ and $L_i = I$ for $i = 1, \dots, k$ we have that: (i) N_i and L_i have full row-rank; (ii) $\sum_{i=1}^k L_i \Theta_i L_i{}^T = \sum_{i=1}^k \Theta_i$ is a diagonal matrix, and its eigenvalues are $\gamma_j = \sum_{i=1}^k \Theta_{ij}$ with eigenvectors $V_j = e_j$ (e_j being the jth column of I), $j = 1, \dots, n'$; (iii) $V = [V_1 \dots V_{n'}] = I$, and then $v = V^T u = u$, i.e., $u_j/v_j = 1$ for $j = 1, \dots, n'$. Therefore, for multicommodity problems bound (12) can be written as

$$\rho \leq \max_{j \in \{1,\dots,n'\}} \frac{\sum_{i=1}^{k} \Theta_{ij}}{\Theta_{0j} + \sum_{i=1}^{k} \Theta_{ij}} < 1, \tag{21}$$

where Θ was defined in (7).

In addition, for multicommodity flows the strong assumption $\hat{u}_j/\hat{v}_j \leq u_j/v_j$ of Proposition 1 is satisfied, since $u_j/v_j = \hat{u}_j/\hat{v}_j = 1$. Therefore bound (21) is effectively reduced, if flows only are regularized, by adding even a small quadratic term $Q_i \succ 0$, $i = 1, \dots, k$, to a linear multicommodity problem. The quadratic term of the regularized algorithm thus guarantees such a reduction. Although a reduction in the bound does not mean a reduction in the spectral radius (which is the instrumental factor), we note that in the last interior-point iterations the spectral radius is always observed to tend to one [7], and so does the upper bound (21). Therefore, a reduction in the bound will also mean a reduction in the spectral radius in these last, numerically expensive for PCG, interior-point iterations. We remark that, although a smaller spectral radius means that theoretically the preconditioner is a better approximation of the inverse of the Schur complement matrix, the practical effect of the regularization—i.e., the reduction in number of PCG iterations performed—has to be computationally tested anyway. It is also worth noting that if N, the node-arc incidence matrix, is a square matrix then

$$\begin{aligned}(D^{-1}(C^T B^{-1} C)) &= \left(\Theta_0 + \textstyle\sum_{i=1}^k \Theta_i\right)^{-1} \left(\textstyle\sum_{i=1}^k \Theta_i N_i^T \left(N_i \Theta_i N_i^T\right)^{-1} N_i \Theta_i\right) \\ &= \left(\Theta_0 + \textstyle\sum_{i=1}^k \Theta_i\right)^{-1} \left(\textstyle\sum_{i=1}^k \Theta_i\right),\end{aligned}$$

which is equal to a diagonal matrix whose jth component is $\left(\Theta_{0j} + \sum_{i=1}^k \Theta_{ij}\right)^{-1} \left(\sum_{i=1}^k \Theta_{ij}\right)$. In this case, (21) does not actually provide a bound, but the true spectral radius. Although problems with N square are not of practical interest (they have at most one feasible solution), it shows how tight the bound is in a limit situation. Another interesting observation from this result is that slacks are instrumental: otherwise Θ_0 would be 0, and the bound would be one, independently of the regularization performed. This suggests that, even for primal block-angular (or multicommodity problems) with equality linking constraints (i.e. saturated arcs) it is worth to consider slacks with negligible upper bounds. This justifies what was empirically observed in [10], where a quadratic multicommodity flow problem—from the statistical disclosure control field— with equality capacity constraints (all arcs were saturated) was solved very efficiently with this algorithm considering slacks with very small upper bounds. Additional arguments on the benefits of the regularization term for decreasing the spectral radius are provided by the computational results of next sections.

4 Approximate sensitivity analysis

The simple expression of bound (21) allows us to perform a local sensitivity analysis on small regularizations. Let us consider that $j \in \{1, \ldots, n'\}$ is the index providing the maximum in (21). By (7), the elements Θ_{ij}, $i = 0, \ldots, k$, are

$$\Theta_{ij} = \frac{1}{Q_{ij} + (U_{ij} - X_{ij})^{-1} W_{ij} + X_{ij}^{-1} Z_{ij}}, \tag{22}$$

where Q_{ij}, U_{ij}, W_{ij}, X_{ij} and Z_{ij} are scalars. For linear problems $Q_{ij} = 0$. Adding a small quadratic regularization $Q_{ij} = \delta_i$, $i = 0, \ldots, k$, both the spectral radius of matrix $D^{-1}(C^T B^{-1} C)$ and the bound (21) will change. Performing an accurate sensitivity analysis on the spectral radius is not possible [18, Section 8.1.2], but it can be done for the bound. Defining $t_i = (U_{ij} - X_{ij})^{-1} W_{ij} + X_{ij}^{-1} Z_{ij} > 0$, $i = 0, \ldots, k$, the bound (21) can be written as a continuous function of $\vec{\delta} = (\delta_0, \ldots, \delta_k)$:

$$f(\vec{\delta}) = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{\delta_i + t_i}}{\displaystyle\frac{1}{\delta_0 + t_0} + \sum_{i=1}^{k} \frac{1}{\delta_i + t_i}}. \tag{23}$$

We next show the effect of regularizing either the flows, slacks, or both of them. Two different regularization schemes are considered: (i) the regularization value is the same for all the flows and/or slacks (i.e., $\delta_i, i = 0, \ldots, k$ is either 0 or δ); (ii) the regularization may be different for any flow and/or slack (i.e., $\delta_i, i = 0, \ldots, k$ may be different). Variants (35) and (37), and variants (36) and (38), of below Subsection 5.4 belong, respectively, to these two different schemes.

(i) The regularization value is the same for all the flows and/or slacks

We first remark that in this case (23) matches the more general expression

$$f(\delta) = \frac{\sum_{i=1}^{k} f_i(\delta)}{\sum_{i=0}^{k} f_i(\delta)}, \tag{24}$$

whose derivative (to be used later) is

$$f'(\delta) = \frac{f_0(\delta) \sum_{i=1}^{k} f_i'(\delta) - f_0'(\delta) \sum_{i=1}^{k} f_i(\delta)}{\left(\sum_{i=0}^{k} f_i(\delta)\right)^2}. \tag{25}$$

Let us first consider the regularization on flows only, thus $\delta_0 = 0$, and $\delta_i = \delta$, $i = 1, \dots, k$. Then, from (25), (23) and its derivative are

$$f_1(\delta) = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{\delta + t_i}}{\displaystyle\frac{1}{t_0} + \sum_{i=1}^{k} \frac{1}{\delta + t_i}}, \qquad f_1'(\delta) = \frac{\displaystyle\frac{-1}{t_0} \sum_{i=1}^{k} \frac{1}{(\delta + t_i)^2}}{\left(\displaystyle\frac{1}{t_0} + \sum_{i=1}^{k} \frac{1}{\delta + t_i}\right)^2}. \tag{26}$$

Since $t_0 > 0$, $f_1'(\delta) < 0$ and $f_1(\delta)$ is a monotonically decreasing function. This holds not only for the jth component associated to the maximum in (21), but for all the components. Therefore the bound is always reduced if flows only are regularized. This is consistent with Proposition 1, which only considered the addition of quadratic costs $Q_i \succ 0$, $i = 1, \dots, k$, with $Q_0 = 0$.

In the second case, if we only regularize the slacks, i.e., $\delta_0 = \delta$ and $\delta_i = 0$, $i = 1, \dots, k$, (23) and its derivative become

$$f_2(\delta) = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{t_i}}{\displaystyle\frac{1}{\delta + t_0} + \sum_{i=1}^{k} \frac{1}{t_i}}, \qquad f_2'(\delta) = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{t_i}}{\left(1 + (\delta + t_0) \displaystyle\sum_{i=1}^{k} \frac{1}{t_i}\right)^2}. \tag{27}$$

Since $t_i > 0$, $i = 1, \dots, k$, $f_2'(\delta) > 0$, and thus $f_2(\delta)$ is monotonically increasing, which means that locally the bound on the spectral radius will get worse.

Finally, the more general case considers a regularization on both the flows and slacks, i.e., $\delta_i = \delta$, $i = 0, \dots, k$. In this case, using again (25), (23) and its derivative are

$$f_3(\delta) = \frac{\displaystyle\sum_{i=1}^{k} \frac{1}{\delta + t_i}}{\displaystyle\frac{1}{\delta + t_0} + \sum_{i=1}^{k} \frac{1}{\delta + t_i}}, \qquad f_3^{'}(\delta) = \frac{\displaystyle\frac{1}{(\delta + t_0)^2} \sum_{i=1}^{k} \frac{(t_i - t_0)}{(\delta + t_i)^2}}{\left(\displaystyle\frac{1}{\delta + t_0} + \sum_{i=1}^{k} \frac{1}{\delta + t_i}\right)^2}. \tag{28}$$

In this case $f_3'(\delta)$ can be either positive or negative depending on $\sum_{i=1}^{k} \frac{(t_i - t_0)}{(\delta + t_i)^2}$. This situation is discussed later, in paragraph containing (32).

(ii) The regularization may be different for any flow and/or slack.

In this case the effect of a particular regularization point $\vec{\delta} = (\delta_0, \delta_1, \dots, \delta_k)$ can be analyzed by studying the directional derivative of $f(\vec{\delta})$ along the non-negative direction vector $\omega = (\omega_0, \omega_1, \dots, \omega_k)$, i.e.,

$$\nabla_\omega f(\vec{\delta}) = \sum_{i=0}^{k} \frac{\partial f(\vec{\delta})}{\partial \delta_i} \omega_i, \tag{29}$$

where

$$\frac{\partial f(\vec{\delta})}{\partial \delta_0} = \frac{\dfrac{1}{(\delta_0+t_0)^2}\displaystyle\sum_{i=1}^{k}\frac{1}{\delta_i+t_i}}{\left(\dfrac{1}{\delta_0+t_0}+\displaystyle\sum_{i=1}^{k}\frac{1}{\delta_i+t_i}\right)^2} > 0, \tag{30}$$

$$\frac{\partial f(\vec{\delta})}{\partial \delta_i} = \frac{-\dfrac{1}{\delta_0+t_0}\,\dfrac{1}{(\delta_i+t_i)^2}}{\left(\dfrac{1}{\delta_0+t_0}+\displaystyle\sum_{i=1}^{k}\frac{1}{\delta_i+t_i}\right)^2} < 0 \quad i=1,\dots,k. \tag{31}$$

From (30) the term $\frac{\partial f(\vec{\delta})}{\partial \delta_0}\omega_0$ of (29) is positive, while from (31) the remaining terms of (29) are negative. Then we have: (i) the bound is always reduced if flows only are regularized; (ii) the bound increases if regularization is only applied to slacks; (iii) if both flows and slacks are regularized, the contribution to the slacks term in (29) increases the bound, independently of the sign of (29); if (29) is negative, it would become more negative if slacks were not regularized. We conclude that the regularization of slacks never improves (i.e., locally never decreases) the bound on the spectral radius.

It is worth noting that the results (26)–(28) for the regularization scheme (i) (i.e., regularization is the same for slacks and/or flows) can be derived from (29)–(31). Indeed, $f_1'(\delta)$ in (26) and $f_2'(\delta)$ in (27) are equal to the evaluation of (29) along, respectively, the directions $\omega = (0,1,\dots,1)$ and $\omega = (1,0,\dots,0)$ when $\delta_i = \delta$. Similarly, when both slacks and flows are regularized, $f_3'(\delta)$ in (28) is equal to the evaluation of (29) along the direction $\omega = e$ (i.e., $\omega_i = 1, i = 0,\dots,k$):

$$\begin{aligned}
\nabla_e f(\vec{\delta}=\delta e) &= \sum_{i=0}^{k}\frac{\partial f(\vec{\delta})}{\partial \delta_i}\\
&= \frac{\frac{1}{(\delta+t_0)^2}\sum_{i=1}^{k}\frac{1}{\delta+t_i}}{\left(\frac{1}{\delta+t_0}+\sum_{i=1}^{k}\frac{1}{\delta+t_i}\right)^2} + \sum_{i=1}^{k}\frac{-\frac{1}{\delta+t_0}\,\frac{1}{(\delta+t_i)^2}}{\left(\frac{1}{\delta+t_0}+\sum_{i=1}^{k}\frac{1}{\delta+t_i}\right)^2}\\
&= \frac{\frac{1}{(\delta+t_0)^2}\sum_{i=1}^{k}\frac{\delta+t_i}{(\delta+t_i)^2} - \frac{\delta+t_0}{(\delta+t_0)^2}\sum_{i=1}^{k}\frac{1}{(\delta+t_i)^2}}{\left(\frac{1}{\delta+t_0}+\sum_{i=1}^{k}\frac{1}{\delta+t_i}\right)^2}\\
&= \frac{\frac{1}{(\delta+t_0)^2}\sum_{i=1}^{k}\frac{(t_i-t_0)}{(\delta+t_i)^2}}{\left(\frac{1}{\delta+t_0}+\sum_{i=1}^{k}\frac{1}{\delta+t_i}\right)^2} = f_3'(\delta).
\end{aligned} \tag{32}$$

Therefore, whether $f_3'(\delta)$ is positive or negative (as we questioned after (28)) is irrelevant: from (32) it is clear that the regularization of slacks always increases the bound on the spectral radius in $f_3'(\delta)$.

The above discussion can be summarized in the following result:

Proposition 3. *For the more general regularization scheme with different δ_i:*

1. *The bound on the spectral radius is locally reduced if only the flows are regularized.*
2. *The bound on the spectral radius locally increases if only the slacks are regularized.*
3. *The local reduction (if any) of the bound on the spectral radius is larger if only the flows are regularized, instead of the flows and slacks.*

Proof. This is immediate from (29) by noting that $\frac{\partial f(\vec{\delta})}{\partial \delta_0}\omega_0 > 0$ and $\frac{\partial f(\vec{\delta})}{\partial \delta_i}\omega_i < 0, i = 1, \ldots, k$. $\square$

According to Proposition 3, the safest option for multicommodity flow problems is to perform a regularization on the flows only. This is the default option in the implementation developed and tested in next Section. It is worth noting, however, that, first, this sensitivity analysis is local; and second, it is only valid for the bound on the spectral radius, not the spectral radius. Therefore, it might happen that for some instances the regularization of both slacks and flows provides better results. This will be observed in some instances of Table 8 of below Subsection 5.4.

5 Evaluating the regularized algorithm

5.1 Problem instances

We considered three kind of problems. They are used both in this Section 5 and next Section 6.

The first type corresponds to the well-known PDS problems [6]. Problems obtained with this generator are denoted as PDSt, where t is associated to the planning horizon in days of a military logistic problem. The PDS instances can be retrieved from `http://www.di.unipi.it/di/groups/optimize/Data/MMCF.html`. The second kind was obtained with the implementation of [15] of the Mnetgen generator [1]. It can be retrieved from the above URL. These instances will be denoted as m'-k-d, where m' is the number of nodes, k the number of commodities, and d is related to the density of the network; the larger d the denser is the network. The last set of instances was obtained with the Tripartite generator and with a variation for multicommodity flows of the Gridgen generator. They are known to be difficult linear programming instances, and interior-point algorithms outperformed simplex variants on them [5, 8]. Five such test examples are available. They can be obtained from `http://www-eio.upc.es/~jcastro/mmcnf_data.html`.

5.2 Implementation details

The original code IPM [7] implementing the specialized interior-point algorithm for multicommodity flows has been extended with the regularized barrier (14). The new code will be denoted as RIPM. RIPM is mainly written in C, with only the sparse Cholesky factorization routines coded in Fortran [22]. RIPM is available from the authors on request. The three main parameters to be adjusted in the algorithm are h, the number of terms (minus one) of the power series (11) considered in the preconditioner; ϵ_0, the initial PCG tolerance requested, which is updated at each interior-point iteration; and Q, the diagonal positive semidefinite regularization matrix of (14). As for IPM, the default values for h and ϵ_0 in RIPM are 0 and 10^{-2} respectively. They have been used in all the computational results of Section 6, excluding some few that are clearly marked. For the third and new parameter, an empirical study—based on the results of Subsection 2.2—has been performed for an appropriate choice of Q; this is shown in below Subsection 5.4.

The termination criterion for RIPM is the same than for IPM, but including the effect of matrix μQ due to the quadratic regularization. The code stops when the current primal and dual feasible point (i.e., it solves (5a), (5b)—replacing Q by μQ—, and (5e)) has a relative optimality gap

$$\begin{aligned} & \frac{\left|c^Tx + \frac{1}{2}x^T(\mu Q)x - \left(b^Ty - u^Tw - \frac{1}{2}x^T(\mu Q)x\right)\right|}{1 + \left|(c^Tx + \frac{1}{2}x^T(\mu Q)x)\right|} \\ = \; & \frac{\left|c^Tx - \left(b^Ty - u^Tw\right) + x^T(\mu Q)x\right|}{1 + \left|(c^Tx + \frac{1}{2}x^T(\mu Q)x)\right|} \end{aligned} \tag{33}$$

below some optimality tolerance (by default 10^{-6}). The numerator of (33) is the (absolute) duality gap; it differs from the duality gap of the original linear problem in the quadratic term $x^T(\mu Q)x$. In practice this value should be negligible, since μ in (5c) and (5d) is close to zero in an optimal solution. Anyway, RIPM checks that

$$\frac{x^T(\mu Q)x}{1 + |c^Tx|} \tag{34}$$

is below some tolerance (by default 10^{-6}) to warn the user if the regularization term $x^T(\mu Q)x$ is large. If this warning appears, the problem could be solved again by reducing or removing the regularization term, or increasing the code tolerances (optimality tolerance and ϵ_0). An automatic reoptimization by the code from the current point, discarding the regularization term, could be another option, but this was not implemented; indeed, (34) was larger than 10^{-6} only in one of the executions of Section 6.

An alternative termination criterion would be to consider the primal feasibility, dual feasibility and relative optimality gap of the original linear problem without the regularization term. The primal feasibility conditions of both the original and regularized problem are the same. The dual feasibility of the original problem $A^Ty + z - w = c$ differs from (5b) in the vector $-(\mu Q)x \leq 0$ (considering the regularization matrix μQ instead of Q). Fixing y, a dual feasible solution for the original problem can be obtained by removing $-(\mu Q)x$ and then either decreasing z or increasing w. Unfortunately this is not always possible: for variables x_i strictly between bounds we have $z_i \approx 0$ and $w_i \approx 0$ near the solution, and thus z_i and w_i can not be changed due to the complementarity conditions. Similarly, if $x_i \approx 0 < u_i$—thus $z_i \geq 0$ and $w_i \approx 0$—and $\mu Q_{ii}x_i > z_i$, it is not possible neither to make zero z_i and to increase w_i, nor to increase w_i, otherwise complementarity conditions would be violated. However, near the optimal solution μQx is expected to be not significant, such that the above adjustments on z and w can likely be performed in practice (though it has to be checked).

Table 1: Comparison of the two stopping criteria

	linear			quadratic		
instance	it.	PCG	CPU	it.	PCG	CPU
PDS1	28	333	0.06	42	551	0.09
PDS5	65	1420	2.01	64	1363	1.91
PDS10	88	3089	11.2	80	2221	8.73
PDS15	95	3500	24.9	84	2587	20.0
PDS20	117	6264	67.0	96	3246	40.7
PDS25	129	6209	102	126	5804	97.6
PDS30	133	6156	151	121	4744	130
32-32-12	38	1150	0.42	33	708	0.29
64-64-12	47	1931	2.00	52	2658	2.45
128-64-12	55	3523	11.1	52	2647	8.75
256-64-12	73	7772	64.2	63	3453	32.9
256-256-12	103	4714	184	99	4218	168
tripart1	133	3304	3.47	142	3899	3.74
tripart2	193	10030	39.4	136	2983	16.2
tripart3	109	5683	53.1	104	6099	53.8
tripart4	125	3557	105	131	4049	112

The two above stopping procedures were implemented and compared. Table 1 shows the results obtained with them. Columns "linear" and "quadratic" correspond to the criteria of the original linear problem and the regularized problem, respectively. For each variant the number of interior point iterations (columns "it."), PCG iterations (columns "PCG") and CPU time (columns "CPU") are reported. The computational environment and the 16 instances tested are the same than will be used later in Subsection 5.4. Details about these instances are provided in Subsection 5.1. From Table 1 it can be seen that, in general, there are not significant differences between the two stopping criteria, though the one that considers the quadratic regularization matrix seems to provide slightly faster executions. This stopping criterion has been used in all the computational results of this work.

5.3 Effect of Q on the number of iterations

According to Proposition 1 and Section 4, the bound on the spectral radius is reduced if a regularization is considered, and we can expect a reduction in the number of PCG iterations needed. On the other hand, by Proposition 2, the diagonal elements Q_{ij}, $i = 1, \dots, k$, $j = 1, \dots, n'$, of the regularization matrix should be less or equal than $1/u_{ij}^2$ to have the same complexity result in number of iterations than the nonregularized interior-point algorithm, u_{ij} being the capacity of arc j for commodity i. The complexity increases for larger Q_{ij} values. In many instances the term $1/u_{ij}^2$ would be very small—almost negligible—, causing no reduction in the spectral radius.

Fortunately, in practice it has been observed that there is wide range of regularization values (much larger than $1/u_{ij}^2$) that maintain the number of interior-point iterations; this number of iterations only increases when large regularizations are used. For instance, let us consider problem PDS1, one of the smallest instances considered, and the simple regularization matrix $Q = \delta I$ for some $\delta \in \mathbb{R}$, $\delta \geq 0$. Figure 1 shows the number of iterations for several δ. Note

Figure 1: Interior-point iterations for instance PDS1 using $Q = \delta I$

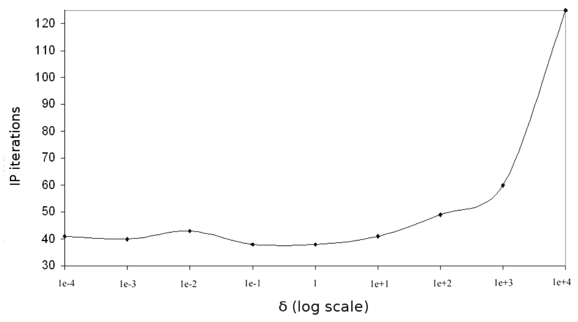
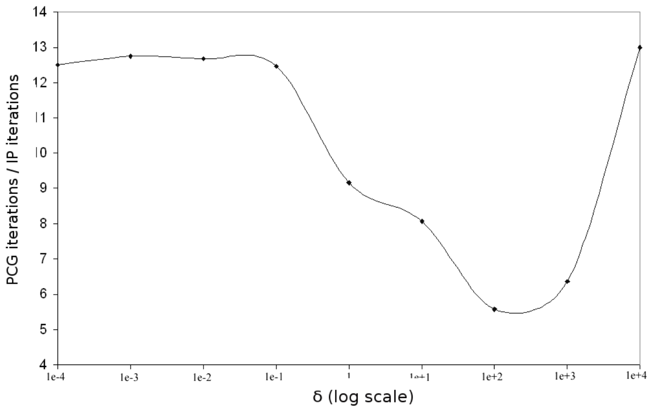


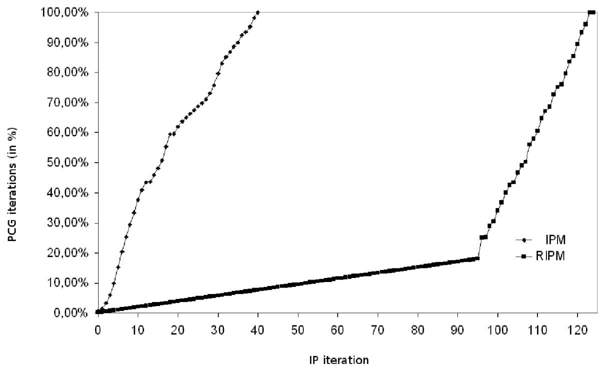
Figure 2: Ratio between PCG and interior-point iterations for instance PDS1 using $Q = \delta I$



that for many arcs of PDS1, the value $1/u_{ij}^2$ was about 10^{-7}, which is much smaller than the values used in Figure 1.

It is also worth noting how Q affects to the number of PCG iterations. Figure 2 shows the average number of PCG iterations needed per interior-point iteration, again for instance PDS1 and the simple regularization matrix $Q = \delta I$ for several δ. It is shown that for small regularizations this average ratio is kept constant, it decreases for δ between 10^{-1} and 10^2, and it significantly increases when $\delta \geq 5 \cdot 10^2$. This does not contradict that the regularization term decreases the number of PCG iterations; indeed, the significant increment of PCG iterations happened in the last interior-point iterations, when the regularization term is very small. This is observed in Figure 3 which shows the evolution of the number of PCG iterations (in percentage of overall PCG iterations) for instance PDS1 using both RIPM (regularized algorithm with rule $Q = \delta I$ and $\delta = 10^4$) and IPM (nonregularized algorithm). As stated before, the number of PCG iterations for RIPM only increased significantly (i.e., with the same slope that for IPM) in last iterations; the overall number of interior-point iterations

Figure 3: Evolution of PCG iterations (in percentage of overall PCG iterations) for instance PDS1



of RIPM was also much larger due to the too large regularization considered. Indeed, this example suggests that such large initial regularizations should be avoided in practice.

5.4 Selection of a static Q

For the selection of a good rule for matrix Q in RIPM, four alternatives were evaluated. According to Proposition 3 these four alternatives only regularize flows. (A comparison between the best of these four regularizations both regularizing only flows, and flows and slacks is presented at the end of this Subsection.) The first regularization alternative, the simplest one, is

$$Q = \delta/\mu_0 I, \tag{35}$$

where $\delta \in \mathbb{R}$ is a positive value, and μ_0 is the value of the centrality parameter at the first iterate: since Q is multiplied by μ, this term guarantees that at the first iteration $\mu Q = \delta I$.

The second variant computed the regularization matrix as

$$Q = \delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}, \tag{36}$$

where the diagonals of $X^{(0)}$ and $Z^{(0)}$ are the starting values of x and z. This choice satisfies that, excluding the upper bounds term of (7), $\Theta = (Q+X^{-1}Z)^{-1}$ will be initially well conditioned (since Q is large when $(X^0)^{-1}Z^0$ is small, and vice-versa).

The third and fourth variants are obtained from (35) and (36) by multiplying them by the iteration counter, i.e., they are, respectively,

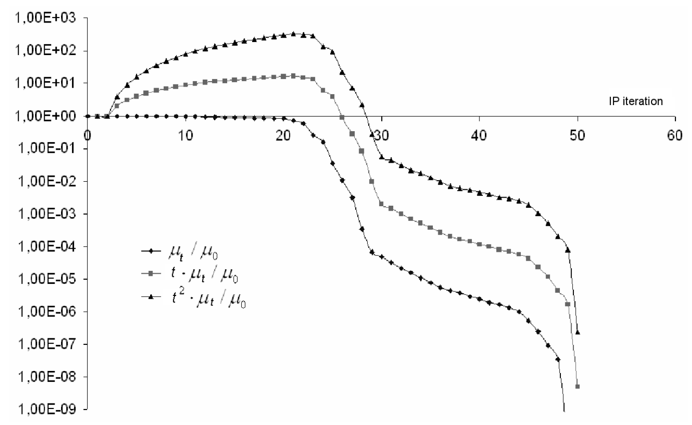
$$Q^{(t)} = t\delta/\mu_0 I, \tag{37}$$

and

$$Q^{(t)} = t\delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}, \tag{38}$$

t being the number of interior-point iteration. Note that the definition of Q changes with t. These two variants are justified because it was observed that

Figure 4: Evolution of μ_t/μ_0, $t\mu_t/\mu_0$ and $t^2\mu_t/\mu_0$ for instance PDS1 and $Q = 1/\mu_0 I$



the effect of the regularization term (which is multiplied by μ) could disappear too early when the solution is being reached (i.e., μ approaches zero). For instance, Figure 4 shows for instance PDS1 and $Q = 1/\mu_0 I$ the evolution of μ_t/μ_0, $t\mu_t/\mu_0$ and $t^2\mu_t/\mu_0$, using a log scale for the vertical axis. Compared to μ_t/μ_0, $t\mu_t/\mu_0$ provides a smoother decrement at last iterations, and it does not result in a very large regularization term, mainly at first iterations, unlike $t^2\mu_t/\mu_0$.

Other variants were tried, but are not reported here since they did not improve the nonregularized algorithm in IPM. One of them, based on Proposition 2, consisted on $Q = U^{-2}$, such that the parameter of the regularized barrier would be 1, the best possible one. In practice, it provided poor results, since in many instances this resulted in a negligible regularization.

The four regularization variants (35)–(38) were implemented and applied to a subset of 16 instances of the PDS, Mnetgen and Tripartite suite. The dimensions of these 16 instances are provided in Table 2: columns k, m' and n' provide the number of commodities, nodes, and arcs, respectively; columns n and m show the overall number of variables and constraints of the resulting linear problem. The four regularizations were tested for 10 different values of $\delta \in \{10^{-8}, 10^{-7}, \ldots, 1, 10^1\}$, and two values for the PCG tolerance $\epsilon_0 \in \{10^{-2}, 10^{-3}\}$. Each resulting combination was also solved with the proximal point regularization barrier problem (13), defining $Q_P = \mu Q$, and computing Q using the four previous regularization variants; that implementation will be denoted by PIPM. This way, RIPM and PIPM are compared under the same conditions, being the only differences the dual feasibility conditions (17) and (18), and the definitions of Θ (19) and (20)—theoretically, there is an important difference: the barrier in RIPM is self-concordant, unlike that of PIPM. This amounts to 2560 executions. Tables 3–6 show respectively for each regularization variant, the best results obtained (i.e., best combination of δ and ϵ_0) for each instance, and for both RIPM and PIPM. Columns "it." and "PCG" report the number of interior-point and overall number of PCG iterations. Columns "CPU" provide the CPU time; all the executions were carried on a Linux SUN

Table 2: Dimensions of the subset of 16 instances

Instance	k	m'	n'	n	m
PDS1	11	126	372	4464	1758
PDS5	11	686	2325	27900	9871
PDS10	11	1399	4792	57504	20181
PDS15	11	2125	7756	93072	31131
PDS20	11	2857	10858	130296	42285
PDS25	11	3554	13580	162960	52674
PDS30	11	4223	16148	193776	62601
32-32-12	32	32	486	16038	1510
64-64-12	64	64	511	33215	4607
128-64-12	64	128	1171	76115	9363
256-64-12	256	64	2320	150190	18030
256-256-12	256	256	2204	566428	67740
tripart1	16	192	2096	35632	5168
tripart2	16	768	8432	143344	20720
tripart3	20	1200	16380	343980	40380
tripart4	35	1050	24815	893340	61565

Table 3: Best results for regularization (35): $Q = \delta/\mu_0 I$

	RIPM					PIPM				
instance	δ	ϵ_0	it.	PCG	CPU	δ	ϵ_0	it.	PCG	CPU
PDS1	10^{-1}	10^{-2}	38	463	0.08	10^{-2}	10^{-2}	39	489	0.08
PDS5	10^{-1}	10^{-2}	59	965	1.54	10^{-1}	10^{-2}	56	863	1.42
PDS10	10^{-2}	10^{-2}	76	1601	7.13	10^{-3}	10^{-2}	77	1536	6.84
PDS15	10^{-2}	10^{-2}	86	2304	19.0	10^{-1}	10^{-2}	89	2481	19.4
PDS20	10^{-1}	10^{-2}	105	4333	49.7	10^{-3}	10^{-2}	107	3877	47.2
PDS25	1	10^{-2}	105	2374	55.6	1	10^{-2}	108	2882	67.7
PDS30	10^{-1}	10^{-2}	113	3050	92.0	10^{-2}	10^{-2}	111	2808	89.6
32-32-12	10^{-1}	10^{-2}	48	2309	0.66	1	10^{-2}	37	1213	0.42
64-64-12	10^{-1}	10^{-2}	63	1445	1.87	10^{-2}	10^{-2}	48	700	1.14
128-64-12	10^{-1}	10^{-2}	70	3793	12.3	10^{-1}	10^{-2}	66	2632	9.19
256-64-12	10^{-2}	10^{-3}	62	3762	34.7	10^{-2}	10^{-3}	62	3964	35.9
256-256-12	10^{-1}	10^{-2}	115	3820	165	10^{-1}	10^{-2}	113	3905	165
tripart1	10^{-3}	10^{-2}	74	3711	2.78	10^{-1}	10^{-2}	63	2682	2.07
tripart2	10^{-1}	10^{-3}	67	2894	11.8	10^{-1}	10^{-2}	72	2368	10.9
tripart3	10^{-1}	10^{-2}	97	5233	47.2	10^{-2}	10^{-2}	83	5490	48.0
tripart4	1	10^{-2}	123	4381	113	1	10^{-2}	127	8313	178

Table 4: Best results for regularization (36): $Q = \delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}$

	RIPM					PIPM				
instance	δ	ϵ_0	it.	PCG	CPU	δ	ϵ_0	it.	PCG	CPU
PDS1	1	10^{-2}	43	506	0.09	1	10^{-2}	41	479	0.09
PDS5	10^{1}	10^{-2}	57	726	1.33	1	10^{-2}	61	962	1.61
PDS10	10^{-2}	10^{-2}	74	1294	6.18	10^{-2}	10^{-2}	73	1481	6.93
PDS15	10^{-1}	10^{-2}	77	1610	14.3	10^{-1}	10^{-2}	89	2326	18.7
PDS20	10^{-2}	10^{-2}	96	2947	38.0	1	10^{-2}	106	4899	56.0
PDS25	10^{1}	10^{-2}	98	1741	44.3	1	10^{-2}	112	3808	73.2
PDS30	10^{-3}	10^{-2}	118	3568	102	1	10^{-2}	118	3852	109
32-32-12	10^{-3}	10^{-2}	39	1179	0.43	1	10^{-2}	40	1523	0.48
64-64-12	10^{-3}	10^{-2}	53	947	1.41	1	10^{-2}	81	5649	4.80
128-64-12	10^{-2}	10^{-2}	60	2304	8.42	1	10^{-2}	68	2968	10.4
256-64-12	1	10^{-2}	86	5149	48.3	1	10^{-2}	96	8886	74.9
256-256-12	1	10^{-2}	113	3774	164	1	10^{-2}	115	3855	167
tripart1	10^{-4}	10^{-2}	53	1646	1.46	1	10^{-2}	71	2440	2.14
tripart2	10^{-2}	10^{-2}	79	3368	13.5	1	10^{-2}	119	11162	36.6
tripart3	10^{-2}	10^{-2}	80	4401	39.6	1	10^{-2}	94	5055	46.5
tripart4	10^{-1}	10^{-2}	131	5835	138	1	10^{-2}	124	7107	153

Table 5: Best results for regularization (37): $Q^{(t)} = t\delta/\mu_0 I$

	RIPM					PIPM				
instance	δ	ϵ_0	it.	PCG	CPU	δ	ϵ_0	it.	PCG	CPU
PDS1	10^{-2}	10^{-2}	43	579	0.11	10^{-2}	10^{-2}	47	563	0.09
PDS5	10^{-2}	10^{-2}	57	911	1.46	10^{-2}	10^{-2}	63	1262	1.82
PDS10	10^{-3}	10^{-2}	78	1784	7.51	10^{-2}	10^{-2}	73	1481	6.58
PDS15	10^{-3}	10^{-2}	84	2494	19.4	10^{-2}	10^{-2}	85	2036	17.0
PDS20	10^{-2}	10^{-2}	106	4260	49.1	10^{-1}	10^{-2}	103	4408	51.8
PDS25	10^{-4}	10^{-2}	114	3929	73.3	10^{-2}	10^{-2}	113	3625	72.2
PDS30	10^{-2}	10^{-2}	120	4275	116	10^{-3}	10^{-2}	115	3235	96.5
32-32-12	10^{-3}	10^{-2}	38	1260	0.42	1	10^{-2}	40	924	0.36
64-64-12	10^{-4}	10^{-2}	54	1173	1.52	10^{-1}	10^{-2}	54	1203	2.11
128-64-12	10^{-2}	10^{-2}	65	2301	9.65	10^{-1}	10^{-2}	64	1736	7.00
256-64-12	1	10^{-2}	85	3108	36.0	10^{-5}	10^{-2}	82	3849	38.5
256-256-12	$5^{(*)}$	10^{-2}	113	2965	140	10	10^{-2}	114	2764	135
tripart1	10^{-1}	10^{-2}	86	1305	1.87	1	10^{-2}	88	1662	2.01
tripart2	10^{-4}	10^{-2}	79	3517	13.9	10^{-2}	10^{-2}	75	3036	12.5
tripart3	10^{-2}	10^{-2}	95	2935	32.6	10^{-3}	10^{-2}	80	4154	38.2
tripart4	10^{-2}	10^{-2}	131	5065	126	10^{-1}	10^{-2}	124	6438	148

$^{(*)}$ using $\delta = 10$ check (34) failed, i.e., $x^T(\mu Q)x/(1 + |c^T x|) > 10^{-6}$

Table 6: Best results for regularization (38): $Q^{(t)} = t\delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}$

	RIPM					PIPM				
instance	δ	ϵ_0	it.	PCG	CPU	δ	ϵ_0	it.	PCG	CPU
PDS1	10^{-3}	10^{-2}	38	472	0.08	1	10^{-2}	42	413	0.08
PDS5	10^{-4}	10^{-2}	56	927	1.47	1	10^{-2}	54	755	1.31
PDS10	10^{-1}	10^{-2}	77	1777	7.44	10^{-2}	10^{-2}	73	1481	6.58
PDS15	1	10^{-2}	81	1607	14.8	1	10^{-2}	84	1961	16.5
PDS20	1	10^{-2}	96	2364	33.9	1	10^{-2}	97	3036	38.6
PDS25	1	10^{-2}	94	1633	42.4	1	10^{-2}	100	2116	49.0
PDS30	1	10^{-2}	99	1667	64.5	1	10^{-2}	121	4388	116
32-32-12	1	10^{-3}	35	1165	0.38	10^{-2}	10^{-2}	41	1137	0.40
64-64-12	10	10^{-3}	45	1235	1.45	10^{-1}	10^{-2}	54	1203	1.54
128-64-12	10	10^{-3}	51	1833	6.79	1	10^{-2}	65	2557	9.06
256-64-12	10	10^{-3}	59	2112	22.8	1	10^{-2}	86	4071	39.8
256-256-12	10	10^{-3}	98	3772	154	1	10^{-2}	110	3354	148
tripart1	1	10^{-3}	142	3844	3.7	1	10^{-2}	86	1907	2.01
tripart2	1	10^{-2}	80	2121	10.5	1	10^{-2}	125	5371	21.3
tripart3	1	10^{-2}	114	1755	28.0	1	10^{-2}	115	7857	66.0
tripart4	0.01	10^{-2}	127	6222	146	1	10^{-2}	149	8191	176

Table 7: Comparison of regularizations: CPU time (seconds)

	Reg. (35)		Reg. (36)		Reg. (37)		Reg. (38)		No Reg.
instance	RIPM	PIPM	RIPM	PIPM	RIPM	PIPM	RIPM	PIPM	IPM
PDS1	**0.08**	**0.08**	0.09	0.09	0.11	0.09	**0.08**	**0.08**	0.09
PDS5	1.54	1.42	1.33	1.61	1.46	1.82	1.47	**1.31**	1.66
PDS10	7.13	6.84	**6.18**	7.92	7.51	6.58	7.44	6.58	7.25
PDS15	19	19.4	**14.3**	19.9	19.4	17	14.8	16.5	21.9
PDS20	49.7	47.2	38	56	49.1	51.8	**33.9**	38.6	56.5
PDS25	55.6	67.7	44.3	73.2	73.3	72.2	**42.4**	49	74.6
PDS30	92	89.6	102	109	116	96.5	**64.5**	116	111
32-32-12	0.66	0.42	0.43	0.48	0.42	**0.36**	0.38	0.4	0.44
64-64-12	1.87	**1.14**	1.41	4.8	1.52	2.11	1.45	1.54	1.49
128-64-12	12.3	9.19	8.42	10.4	9.65	7	**6.79**	9.06	13.2
256-64-12	59.6	47.1	35.9	74.9	36	38.5	**22.8**	39.8	62.4
256-256-12	165	165	164	167	158	**135**	154	148	203
tripart1	2.78	2.07	**1.46**	2.14	1.87	1.01	3.7	2.01	1.7
tripart2	15.5	10.9	13.5	36.6	13.9	12.5	**10.5**	21.3	17.3
tripart3	47.2	48	39.6	46.5	32.6	38.2	**28**	66	62.4
tripart4	**113**	178	138	153	126	148	146	176	265
Sum	643	694	609	764	647	629	**538**	692	900

Reg. (35): $Q = \delta/\mu_0 I$
Reg. (36): $Q = \delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}$
Reg. (37): $Q^{(t)} = t\delta/\mu_0 I$
Reg. (38): $Q^{(t)} = t\delta/\mu_0 X^{(0)}(Z^{(0)})^{-1}$
No Reg.: $Q = 0$

Table 8: Comparison of regularization (38) for only flows vs. flows and slacks, using $\epsilon_0 = 10^{-3}$ and $\delta = 1$ (unless otherwise stated)

	RIPM (flows)			RIPM (flows+slacks)			PIPM(flows)			PIPM(flows+slacks)		
instance	it.	PCG	CPU	it.	PCG	CPU	it.	PCG	CPU	it.	PCG	CPU
PDS1	42	551	0.09	41	511	**0.08**	37	437	**0.08**	37	442	**0.08**
PDS5	64	1363	1.91	64	1340	1.93	56	1183	**1.66**	57	1239	1.71
PDS10	80	2221	**8.73**	82	2432	9.34	76	2519	9.19	75	2501	9.27
PDS15	84	2587	20.0	85	2560	**19.8**	84	3385	23.3	88	3740	25.3
PDS20	96	3246	**40.7**	95	3638	43.4	101	5018	54.5	103	5083	55.3
PDS25	126	5804	97.6	123	4883	**91.3**	117	7257	120	111	5663	92.3
PDS30	121	4744	130	134	7122	165	116	6460	147	113	5285	**128**
32-32-12	33	708	**0.29**	45	1827	0.59	32	860	0.32	35	1216	0.40
64-64-12	52	2658	2.45	50	1939	1.98	50	1812	1.88	47	1412	**1.59**
128-64-12	52	2647	**8.75**	54	3574	11.0	52	2668	8.77	53	2910	9.36
256-64-12	63	3453	**32.9**	62	3476	**32.9**	62	3641	33.7	64	4437	39.5
256-256-12	99	4218	**168**	101	4610	180	101	4310	171	101	4513	176
tripart1	142	3899	3.74	> 400	—	—	95	2894	**2.73**	> 400	—	—
tripart2	136	2983	**16.2**	> 400	—	—	120	7821	27.3	> 400	—	—
tripart3 $^{(*)}$	104	6099	**53.8**	112	6084	55.0	82	9406	71.3	82	10284	76.7
tripart4 $^{(*)}$	131	4049	112	126	3307	**99.9**	126	10006	192	141	11870	225

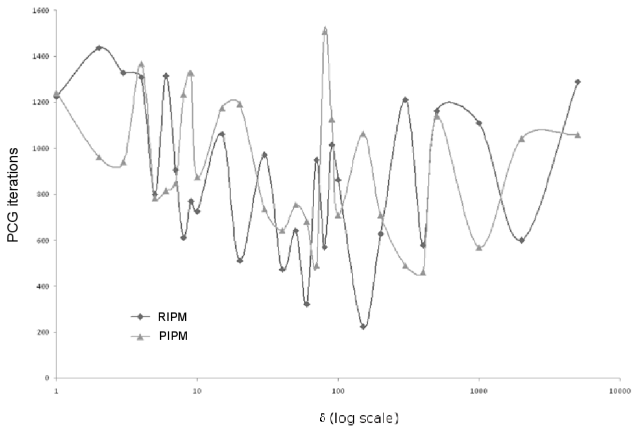
$^{(*)}$ $\delta = 0.1$

Fire V20Z server, credited of 367 Mflops, with two AMD Opteron 2.46GHz processors and 8 GB of RAM (multiprocessor capabilites were not exploited in these runs).

Looking at Tables 3–6 there is not a definitive best approach: neither RIPM nor PIPM always outperformed the other approach; and any of the four regularizations was the most efficient choice for some instance. To have a clearer picture, the results of Tables 3–6 are summarized in Table 7. Table 7 shows the CPU time of the best variant for both RIPM and PIPM, and the CPU of the nonregularized algorithm IPM obtained by setting $Q = 0$. The fastest execution is marked in boldface. Last row reports the total time for all the instances. It is clearly shown that the regularization that provided more "fastest executions" is (38); it is also the variant with the minimum total CPU time. This was chosen as the default option in RIPM for the computational results of Section 6.

Proposition 3 showed the bound on the spectral radius is more effectively reduced when only flows are regularized, instead of flows and slacks. To check this assertion, Table 8 shows the results with (38) for RIPM and PIPM, both regularizing only flows, and flows and slacks. Default parameters $\epsilon_0 = 10^{-3}$ and $\delta = 1$ were used for all the executions, but for two clearly marked instances. The fastest execution is marked in boldface. It can be seen that the option with more fastest executions was RIPM regularizing only flows. In some cases the regularization of slacks and flows provided better results, although these cases are very few and the difference was not significant. This does not contradict Proposition 3, since it only deals with the bound on the spectral radius, but not the spectral radius. However, in general, it seems to be more effective to regularize only flows. In particular, regularizing flows and slacks caused the failure of two instances ("tripart1" and "tripart2"), while the regularization of flows was a more robust option.

Figure 5: PCG iterations of RIPM and PIPM for different δ, in problem PDS5



This section is concluded by noting that, from previous tables, and for these multicommodity instances, the quadratic self-concordant regularization in RIPM was slightly more efficient than the proximal point one in PIPM; and both of them outperformed the nonregularized algorithm. Although RIPM seems to be more efficient for larger problems, in general, however, there is not a significant difference between the two regularized approaches, and the results may be different by tuning some of the parameters. For instance, looking at the number of PCG iterations required for RIPM and PIPM for the particular instance PDS5, for different δ values ranging from 0 to 5000, the results of Figure 5 are obtained. The (unexpected) oscillatory behaviour of RIPM and PIPM shows there is not a definitive better approach, though it is worth noting that the minimum and maximum number of PCG iterations are achieved by RIPM and PIPM, respectively.

5.5 Dynamic Q

Regularizations (35)–(38) do not fully exploit the expression (21) and the sensitivity results of Section 4. The following dynamic regularization, which takes as a basis the best static regularization (38), has been successfully tried:

$$Q_{ij}^{(t)} = t\beta_{ij}\delta/\mu_0 X_{ij}^{(0)}(Z_{ij}^{(0)})^{-1} \; j = 1,\ldots,n' \quad i = 1,\ldots,k. \tag{39}$$

Expression (39) differs from (38) in the factor $\beta_{ij} \geq 1$, which is > 1 if j is the arc that provides the maximum in (21) (i.e., regularization is increased in this arc in an attempt to reduce the spectral radius), and it is 1 for the other arcs. For the arc j' associated to the maximum in (21), $\beta_{ij'}$ was heuristically set as

$$\beta_{ij'} = \min\left\{10, \left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k)}}\right| / \left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k-1)}}\right|\right\},$$

where $\left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k)}}\right|$ and $\left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k-1)}}\right|$ are, respectively, the greater and second greater partial of (31) for this arc (in absolute value). It aims at increasing the regu-

Table 9: Instances dimensions

Instance	k	m'	n'	n	m
PDS10	11	4792	1399	53526	16192
PDS20	11	10858	2857	121137	33115
PDS30	11	16148	4223	180027	48841
PDS40	11	22059	5652	245848	65360
PDS50	11	27668	7031	308281	81263
PDS60	11	33388	8423	371945	97319
PDS70	11	38369	9750	427663	12546
PDS80	11	42472	10989	472863	126539
PDS90	11	46161	12186	513635	139899
128-64-10	64	128	1182	76566	9046
128-64-11	64	128	1201	77786	9050
128-128-12	128	128	1204	155044	17188
256-64-10	64	256	2336	151293	18109
256-64-11	64	256	2334	151154	18098
256-64-12	64	256	2320	150190	18030
512-128-12	128	512	4786	616189	68989
512-256-12	256	512	4810	1234949	134405
512-512-12	512	512	4786	2454022	265222
tripart1	16	192	2096	35632	5168
tripart2	16	768	8432	143344	20720
tripart3	20	1200	16380	343980	40380
tripart4	35	1050	24815	893340	61565
gridgen1	340	1025	3072	986112	331072

larization when $\left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k)}}\right|$ is large, dividing by $\left|\frac{\partial f(\vec{\delta})}{\partial \delta_{(k-1)}}\right|$ to obtain a relative value greater than one. A maximum of 10 was set to avoid very large regularizations, which would increase the number of interior-point iterations. For this same reason, $\beta_{ij'}$ was set to one in the last interior-point iterations. In general, as it will be shown in the computational results of Section 6, this dynamic regularization scheme outperformed the static regularization (38).

6 Computational results

From the empirical analysis of Section 5, for the computational results we have considered RIPM with the regularized version (38) and its dynamic variant (39), and $\epsilon_0 = 10^{-2}$ as default options. For the static regularization (38) the parameter δ was set to 1 for the Mnetgen and Tripartite/Gridgen instances, while it was 0.1 for the PDS ones. For the dynamic regularization (39), δ was 50 for the Mnetgen instances, and was 1 for the remaining ones. These default settings have been used for all the runs of this Section, unless otherwise stated for a couple of executions (due to numerical issues associated to PCG). Note that tuning those parameters it is possible to obtain better results (as in Section 5). However, the purpose of this Section is to show that RIPM with default values may be an efficient interior-point approach for some multicommodity flows, even more than the nonregularized algorithm. Executions were performed on the same machine used for the computational results of Section 5.

Table 9 reports the dimensions of the PDS, Mnetgen and Tripartite/Gridgen instances tested. Table 10 shows the results obtained with IPM, RIPM and the dual simplex and barrier algorithms of CPLEX-11. The hybrid and primal sim-

Table 10: Results with IPM, RIPM (regularizations (38) and (39)) and CPLEX-11 (dual and barrier)

	IPM			RIPM reg. (38)			RIPM reg. (39)			CPLEX dual		CPLEX barrier	
Instance	it.	PCG	CPU	it.	PCG	CPU	it.	PCG	CPU	it.	CPU	it.	CPU
PDS10	80	3830	12.2	86	2964	10.9	77	2248	*8.67*	5116	**0.88**	33	7.07
PDS20	106	11011	100	100	3667	44.9	89	3079	*39.1*	17830	**7.37**	39	30.76
PDS30	126	9717	204	114	4287	114	113	3926	*106*	29336	**17.27**	38	123.76
PDS40	135	12700	469	135	9261	367	133	5902	*266*	49160	**46.25**	38	191.87
PDS50	135	13603	718	138	8707	*518*	177	17619	924	62983	**62.34**	38	297.66
PDS60	135	19264	1350	147	15577	1170	173	12359	*1100*	73610	**76.50**	40	485.10
PDS70	150	18526	1830	158$^{(1)}$	9998	*1230*	184	15496	1700	97271	**99.46**	38	645.24
PDS80	152	19701	2340	162	10360	*1570*	185	14513	2110	117281	**138.36**	40	719.63
PDS90	149	18644	2560	171	13207	*2180*	190	17109	2730	118797	**139.14**	40	814.56
128-64-10	70	4953	14.6	69	4441	13.4	53	1038	**4.94**	17686	12.98	18	55.92
128-64-11	76	5481	16.5	64	3020	10.3	57	928	**5.08**	19295	11.72	18	57.97
128-128-12	97	3839	26.8	90	2779	21.2	64	730	**9.49**	39987	33.83	21	145.6
256-64-10	90	5770	54.3	85$^{(2)}$	5204	50.4	67	1292	**19.4**	36462	40.05	18	360.34
256-64-11	83	4748	46.3	73	6467	59.9	56	973	**18**	35246	31.27	17	362.11
256-64-12	132	87698	620	61	3018	**29.5**	94	4174	42	36193	33.96	11	302.49
512-128-12	117	6164	396	113	4105	*344*	115	7166	415	98856	**88.27**	—	>3000
512-256-12	139	6945	883	139	6433	829	121	5566	*761*	181380	**157.3**	—	>3000
512-512-12	179	12074	2760	163$^{(2)}$	4782	1560	133	1504	*801*	342622	**399.0**	—	>3000
tripart1	58	1976	1.7	74	587	*1.26*	105	1959	2.33	4197	**1.12**	21	3.99
tripart2	87	4092	17.3	142	2299	**14.7**	149	3525	18.5	58316	106.97	25	36.01
tripart3	90	6978	62.4	146	3236	**42.3**	105	3133	43.8	96592	382.47	28	138.8
tripart4	133	14660	265	151	2405	96.8	125	1731	**77.9**	165668	1638.12	29	1323.2
gridgen1	242	96877	7400	241	31280	2420	182	11691	**1080**	—	>15000	64	12288

(1) check (34) failed, i.e., $x^T(\mu Q)x/(1+|c^T x|) > 10^{-6}$

(2) $\delta = 2$

plex CPLEX-11 variants were also tried, but they never improved either the dual or the barrier, so their results are not reported. Like for RIPM, default settings were used for CPLEX-11, including the automatic selection of the barrier ordering scheme, which is one of the most instrumental parameters of the interior-point algorithm; indeed, results did not improve by manually setting either the minimum degree or nested dissection ordering schemes. The meaning of the columns is the same than in previous tables. The fastest of the five executions for each instance is marked in boldface. If RIPM did not provide the fastest execution, then the fastest of its two regularization variants is marked in italic.

From Table 10, the dynamic regularization (39) of RIPM was more efficient than the static scheme (38) in 14 of the 23 instances. The dynamic regularization was significantly more efficient in the largest Mnetgen and Tripartite/Gridgen instances, while it was outperformed by the static regularization in the largest PDS instances. Table 10 also clearly shows that RIPM was more efficient than IPM in all the instances. The efficiency is more notable in the Tripartite/Gridgen instances, where for the two largest problems RIPM was about three and seven times faster. In some instances the benefit added by the regularization term to the solution of systems with PCG is substantial: in the largest Mnetgen instance 512-512-12, IPM required an average number of 67 PCG iterations per interior-point iteration, while RIPM with the dynamic regularization only needed 11.

It is known than interior-point methods are not the best approach for PDS and Mnetgen instances. This is clearly observed in Table 10 were the dual simplex was the most efficient option in most instances. For the PDS instances not only the dual, the fastest CPLEX-11 option, outperformed RIPM, but also the generic barrier did. This can be explained by the highly efficient ordering and factorization routines in CPLEX-11. For the Mnetgen instances, the dual simplex is also the fastest CPLEX-11 option. However, in those problems the barrier solver is significantly slower than RIPM (an academic code with standard factorization routines), specially for the larger instances, which exceeded a time limit of 3000 seconds. On the other hand, the Tripartite/Gridgen instances are known to be difficult instances for simplex-like methods, and interior-point algorithms outperform them. It can be seen that the barrier solver was by far the most efficient CPLEX-11 approach. However, RIPM (and also IPM) was significantly faster than the CPLEX-11 barrier. As far as we know, up to now IPM was the most efficient algorithm for these difficult instances [8]; this no longer holds, since RIPM is a more efficient approach. This result shows that RIPM may be a promising approach for the solution of nonlinear smooth convex separable multicommodity flow problems. Indeed, for these problems simplex-like algorithms are not competitive against interior-point algorithms, and RIPM seems to compare very well against generic state-of-the-art barrier solvers. The regularization of a convex nonlinear objective function in RIPM may be used to gurantee strict convexity, thus improving the quality of the preconditioner.

As stated above, the results for RIPM were obtained with default options, since this was the purpose of this Section. However, we note they are not the best results that can be obtained with RIPM; especially for the larger instances, there is room for improvement by tuning the parameters. For instance, problem PDS60 could be solved in 147 iterations, 5954 PCG iterations and 610 seconds (instead of the 1100 seconds of Table 10); and problem PDS90 could be solved in 140 iterations, 6018 PCG iterations and 1280 seconds (instead of the 2180 seconds of Table 10). However, even with these significant improvements, RIPM is still outperformed by CPLEX-11. On the other hand, problem “gridgen1” was solved in 219 iterations, 5703 PCG iterations and 618 seconds, reducing the 1080 seconds of Table 10, thus making it even more competitive against general solvers like CPLEX-11.

7 Conclusions

From the results of this work, it is clear that the new regularized version outperforms the specialized interior-point method for multicommodity flows implemented in IPM. This means that linear multicommodity flow problems are more efficiently solved by specialized interior-point methods based on PCG, if they are dealt with as a sequence of quadratic multicommodity flow problems. However, for some standard classes of multicommodity flow problems, as the PDS and Mnetgen ones, dual simplex algorithms are still more efficient than the new regularized approach. On the other hand, for some classes of difficult multicommodity problems, interior-point methods outperform simplex variants; for those instances, the regularized specialized interior-point algorithm could be considered one of the most efficient available approaches. The application of the regularized algorithm to nonlinear smooth convex separable multicommodity flow problems is part of the future research we intend to pursue.

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